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Student performance and attitudes in a collaborative and flipped linear algebra course

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Flipped learning is gaining traction in K-12 for enhancing students' problem solving skills at an early age; however, there is relatively little large-scale research showing its effectiveness in promoting better learning outcomes in higher education, especially in mathematics classes. In this study, we examined the data compiled from both quantitative and qualitative measures such as item scores on a common final and attitude survey results between a flipped and a traditional Introductory Linear Algebra class taught by two individual instructors at a state university in California in Fall 2013. Students in the flipped class were asked to watch short video lectures made by the instructor and complete a short online quiz prior to each class attendance. The class time was completely devoted to problem-solving in group settings where students were prompted to communicate their reasoning with proper mathematical terms and structured sentences verbally and in writing. Examination of the quality and depth of student responses from the common final exam showed that students in the flipped class produced more comprehensive and well-explained responses to the questions that required reasoning, creating examples, and more complex use of mathematical objects. Furthermore, students in the flipped class performed superiorly in the overall comprehension of the content with a 21% increase in the median final exam score. Overall, students felt more confident about their ability to learn mathematics independently, showed better retention of materials over time, and enjoyed the flipped experience.

Keywords: flipped classroom; flipped learning; linear algebra; student attitudes toward mathematics; inverted learning; collaborative learning; higher education

1. Introduction and Background

Although some of the practices of flipped learning, sometimes called inverted learning, have been in use for decades, it is a pedagogical practice that came into the public eye in recent years and has been gaining traction in many disciplines at several levels of education. The specifics of flipped learning can vary widely from class to class, but the basic principle lies in the name: the structure of learning is flipped upside-down, with initial exposure to material (traditionally taking place during a lecture) taking place independently; more active problem-solving, practice with the material, and higher-level thinking (traditionally homework) takes place in the classroom. What has allowed for such an increase in momentum for flipped learning is the advent and increased accessibility of new technologies that allow for initial exposure to be moved online and outside of the classroom. Strayer proposed a standard for what determines a flipped class, as opposed to a traditional class that utilizes technology, which we saw used in several other studies: a modern flipped classroom includes the “regular and systematic use of interactive technologies in the learning process”. [1] The initial exposure in a flipped class can be through a textbook, video lectures made by the class instructor, videos posted online by other creators like Khan Academy, interactive educator services such as ALEKS, or a combination of the above.

In-class activities vary as much as the at-home lecture, depending on the subject and

what the instructor chooses to do with class time; what matters is that class time is interactive and engaging, and the instructor behaves more as a facilitator than a lecturer. Kim et al. did research on three variations of the flipped classroom format in three disciplines and found the advantages and pitfalls of each. Specifically, Kim et al. investigated the relationship of different technologies to the flipped learning experience by collecting data on student and instructor perceptions of each flipped experience. Nine design principles were developed for flipped classrooms, most of which “appear also to apply to a typical undergraduate face-to-face course”. [2] These include mechanisms for motivating students, methods of evaluating student learning, and advice on the respective roles of teacher and student in the flipped classroom. Most of the recommended design principles center around the significance of engagement, creating a class structure that is clear and cohesive, and ensuring an equal accessibility to materials for all students.

Research has also been done on whether student and instructor perceptions show that flipped class formats are more favorable than other formats. Davies et al. compared three formats in the same course: a traditional class, a flipped class, and a simulation class which was technology-based. They found that the flipped class was equally or more effective as the other two in “delivering the class” and more scalable than either. [3]

Strayer found through a survey and interviews on student perceptions that students preferred the structure of the traditional class to the flipped class because they reported that the flipped class was less engaging. However, Strayer’s implementation of the flipped class was not executed in alignment with many concepts behind the flipped class. Though the paper does not specify the flipped class’s formatting, there are mentions by Strayer and quotes from students indicating that some material was still lectured in class, which means that while Strayer’s definition above is satisfied, the class was not fully flipped. Additionally, students in an interview are reported as agreeing that “most students felt lost and did not know what was happening in class. These statements are examples of the unsettled feelings caused partly by such varied activity in the classroom”. [1] These facts from Strayer’s research indicate that although his implementation of a flipped classroom was unpopular with students, it is possible that the same would not be true of a class implemented in a more predictably structured, fully-flipped way.

Overall benefits of the flipped learning experience have been reported to be the ability of students to determine their own pacing, rewind and re-watch video lectures as needed, and watch in a time and place that is convenient for them. [2, 3] If time in class is replaced by meaningful activities, the increased interaction between students that happens in the interactive classroom activities also helps students solidify their understanding and help one another. In a flipped class, the instructor can give more differentiated, individual attention to those students who need it, because their time is spent facilitating activities rather than at the front of the room asking questions to which only strong students tend to respond. [3, 4] Other benefits found by Love et al. include a stronger belief by students in a flipped class that the subject matter was important to their careers, and most said that the nature of the flipped class (specifically explaining ideas to peers and working in groups on the board) helped them understand the material. [5]

The same article also mentions that flipped learning could be an effective way to keep STEM students engaged and keep them in their majors. The importance of engagement specifically in linear algebra is also discussed by Chang [6], who recommends prefacing technical exposure with an application idea to keep students engaged and remind them of the real-life practicality of the linear algebra they are learning. Bringing back this application at the end of a unit can reinforce this enthusiasm and create a more full-circle, rounded experience.

Though anecdotal evidence like the above has suggested that the flipped classroom is successful in helping student learning, comprehensive research is still needed to study how

flipped learning affects specific student learning outcomes such as performance on exams, the quality of response students are able to produce, and the depth of their thoughts and understanding as demonstrated by those responses. Even in those studies which looked at data on students achievement and demonstration of learning objectives, the sample size was very small or the performance data was not a focus of the study.[3, 7] This study aims to look at data on student perceptions and attitudes as well as quantitative and qualitative data on student performance in two comparable Introductory Linear Algebra courses. These findings aim to help educators considering a flipped class format to see the ways in which this flipped implementation helped student learning and ways in which the format could be improved to help students even further.

2. Implementations and Methods

Both flipped and traditionally-taught classes used the same textbook and covered the same topics. Both classes meet twice a week, for 75 minutes each time. Both classes were taught by full-time tenured professors who have extensive experiences teaching undergraduate lower-division mathematics classes.

2.1. *Logistics of the Flipped Classroom*

In the research presented here, the initial exposure to novel content in the flipped class was in the form of 15 to 25 minute video lectures recorded by the professor, often split up into even shorter videos by topic. These videos were recorded and administered on Panopto, a service connected to the University's Learning Management System. After watching and taking notes on the assigned videos at home, students took a short online quiz prior to their class meeting as a check for comprehension (or a ticket-at-the-door). Students also had access to Piazza, an online collaboration forum to allow students to post and discuss questions asynchronously. The instructor can then endorse student answers or post his or her own responses. This allows for student questions on videos, assigned homework, or other assignments to be answered quicker, rather than having the student wait until the next class meeting or office hours.

In class, students worked in groups of 4 to 5 on a group quiz, which was written by the instructor with the intent to be more difficult than something students could accomplish successfully on their own as they would do with homework outside of a traditionally formatted class. The groups were assigned by the instructor at the start of the term using responses from a survey. Groups are meant to be heterogenous with respect to students' fields of study, familiarity with technology, degrees of confidence in mathematics, and background knowledge gained from previous courses. With the exception of slight changes due to occasional dynamic conflicts, the makeup of the groups stayed the same throughout the course of the term.

Students took the group quiz in an Active Learning Classroom with access to the textbook, Internet, and white board walls, while being encouraged to utilize other groups and the professor to work out solutions. Group quizzes were more focused on the process behind linear algebra concepts and gaining a deep understanding of material than on the actual solutions on the paper handed in by students. Though the group quiz grades did count toward students overall grades, solutions were often checked and worked through with the instructor before the quiz was handed in. In addition to online and group quizzes, students handed in assigned homework problems from the textbook and took three midterm exams and a final exam.

Toward the end of the term, students worked with the same groups from the daily

quizzes to create an original project on an applied topic related to linear algebra that was often driven by real-world experiences. This project was completed entirely outside of class and was then presented in a poster-presentation-like atmosphere to their instructor and peers. Learning objectives and benefits of this project included the experience of applying newly learned linear algebra skills to real, open-ended problems that might require many steps to solve; practicing an experience like one students might experience in the workforce and developing skills in teamwork and communication; publicly speaking about a difficult topic in a way that laymen can understand; and learning to use software like Google Hangout for collaboration, Google Docs for real-time edits and document sharing, and Microsoft PowerPoint for creating a professional poster.

2.2. *Logistics of Traditional Classroom*

The traditional class, discussed in this study for comparison, was taught by a different tenured faculty at the same university. The class was taught on different days of the week, but each met for 75 minutes at 9:30 a.m. twice weekly. The instructor of the traditionally formatted class would lecture most of the time, except for approximately twenty minutes a week of group work with groups of 3-5 students. In these groups, students would work on assigned problems together. However, each student in the group would hand in his or her own paper, so there was no group grade component. Homework was collected about every 1.5 weeks in the traditional class. Students also took a total of five quizzes in the semester, which were administered as traditional, independent quizzes for the first 20-30 minutes of class. The instructor incorporated a few questions on the quizzes and the two midterms that were similar in style to the flipped class, to expose students to questions similar to the final exam. Students in this class did not do any project or presentation.

2.3. *Assessment*

Students in the traditional class had two midterm exams and flipped students had three. The second traditional midterm was very similar to the third flipped midterm, but the other midterms were not the same. Both classes took a common final which were re-graded by two individuals using a common rubric before computing the scores discussed in the findings. The final was written by the flipped instructor, and the traditional instructor tried to work a few problems written by the flipped instructor into traditional quizzes so that students could have exposure to that style of testing. The final exam consisted of five parts: definition, computation (split into two sections), true/false in the form of multiple choice, applications, and construction. The flipped instructor's prior experiences have indicated that students in this class come in less prepared to perform tasks that are higher up on the Bloom's Taxonomy [8] such as reasoning and providing one's own example. Hence materials are designed to allow students to practice and get better at those areas. In this exam, for instance, students have historically performed better at computation and applications (which are computational in nature), which are skills ranked lower in Bloom's pyramid; thus, the areas in which we hoped they would improve are true/false, definitions, and construction.

2.4. *Demographics*

Both the flipped and traditional classes took place in fall 2013 at an urban, comprehensive State University in California. The university's 36,000+ population consists of over 90% commuting students. Majors of students in the classes being studied are displayed in Table 1.

Table 1. The number of students in each major at census in each class. Note that some similar majors were combined into one field for simplicity. For example applied math, pure math, and math education were combined to make the field “Mathematics BS or BA.”

Major	Flipped	Traditional
Aerospace Engineering BS	1	0
Applied Statistics MS	0	1
Post-Baccalaureate	2	2
Chemical Engineering BS	1	1
Chemistry BS	1	0
Computer Science BS	15	8
Construction Engr Mgmt BS	0	1
Electrical Engineering BS	1	2
Geology BS	1	0
Mathematics BS or BS	10	15
Mechanical Engineering BS	2	3
Physics BS	3	4
Sociology BA	0	1
Undeclared Undergraduate	0	2
TOTAL	37	40

The only notable difference is the distribution between math and computer science majors; otherwise the mix between the majors is relatively evenly distributed among different majors. Since there is no overwhelming difference, there is no reason to anticipate a large disparity between performances by students based on their chosen field of study. It is also worth mentioning that this study is primarily observational, not a designed experiment. Individuals were not randomly assigned to teaching modality, and conditions were not controlled to isolate the effect of the modality, i.e., there could be lurking variables. In spite of this, we can be confident in the reliability of the comparisons stated in this paper, for the reasons stated previously.

The website students use to register for courses gave no indication that one class would be flipped, so students could not have reasonably anticipated this and registered for one section of linear algebra over another based on a preference. All students in the flipped class signed IRB clearance paperwork; no paperwork was given to the traditional class because students are not discussed individually or quoted in this paper. Between the census, taken at the end of the add/drop period, and the final exam, five students (13.5%) in flipped and six (15%) in the traditional class either withdrew or stopped coming to class and did not take the final.

3. Findings

3.1. Cumulative Final Exam

Since both classes took an identical, cumulative final exam, it is helpful to examine student performance on two levels: first how student scores compared on the exam as a whole, and second how the quality of student responses compared on key questions. By looking at actual responses of students in open-ended questions, we are able to see how well students in each class could not only find but effectively communicate the correct answer. To evaluate the score performances of students in the two classes, two graders (one professor and one student assistant) re-graded all of the responses using the same rubric for both classes. They first graded for points, then went back through two chosen questions to find student errors made repeatedly and tally the number of students who made each type of mistake. The results of the analysis of the two chosen questions are discussed in the subsequent section, and the entire final exam is available in Appendix A.

Table 2. Distribution of scores in each class on the common final exam.

Letter	Flipped	Traditional
A (135-150 pts)	2	0
B (120-134 pts)	9	0
C (105-119 pts)	6	3
D (90-104 pts)	4	6
F (0-89 pts)	11	25
TOTAL	32	34

Table 3. Median and mean scores on each part of the comprehensive final exam in each class. Part 1, Definitions; Part 2.1, Computations; Part 2.2 Computations; Part 3, True/False; Part 4, Applications; Part 5, Construction. For example, students from the flipped class scored an average of 5.25 points on the definition problem (52.5%) while the students from the traditional class scored an average of 1.38 points (13.8%) on the same problem.

Median (Mean)	Part 1	Part 2.1	Part 2.2	Part 3	Part 4	Part 5	Total
Possible points	10	25	40	30	20	25	150
Flipped	5.5 (5.25)	21.5 (19.9)	36 (31.2)	20 (17.8)	15 (12.7)	14 (14)	111 (101)
Traditional	1.5 (1.38)	16 (16.6)	32 (31.1)	10 (9.56)	16 (14.8)	8 (9.06)	79 (82.4)
<i>p</i> -values	< .001	< .012	.146	< .001	.715	.496	.012

When the common final exam was graded using this method, there was a 12.67% increase in mean score from traditional to flipped (a difference of 19 points of a possible 150), and a 32-point or 21.33% increase in median scores. As seen in Table 2, students in the flipped class performed considerably better on the final exam when both sets of exams were graded by the same graders using the same rubric. *p*-values are based on the nonparametric Mann-Whitney U test, since most scores failed Shapiro’s test for normality. Although it is likely that some of this disparity is due to flipped students’ elevated exposure to the style of questions, it can be seen with a closer look at individual student responses, discussed in a subsequent section, that most of the difference comes from student retention and depth of understanding, as opposed to a lack of familiarity with the question type.

Table 3 shows how student performance varied in different skills. The traditional course here is representative of previous traditional courses, with students performing best in computation and application problems. Flipped students also performed well in all of those categories, but were additionally able to far outperform their traditionally taught counterparts in Definitions, True/False, and Construction, which have been historically more difficult for students to master. As will be seen in the deep analysis of student responses in the next section, flipped students’ ability to work out better responses to more difficult types of questions shows a much deeper knowledge. For instance, the higher score in Part 5, Construction, indicates that flipped students demonstrated much better performance on a higher level of Bloom’s Taxonomy than traditional students. Similarly, flipped students’ average scores were almost double on the true/false questions, in which students had to choose a true or false statement from a list of three or four. This requires critical thinking, knowledge of the intricacies of the rules and definitions of linear algebra in order to differentiate common misconceptions from facts.

3.2. Specific Problem Analysis

By taking a closer look at students' written responses on multiple problems on the common final, we can see how students in each class respond differently to questions that are designed to test deeper understanding of concepts. Part 5, Construction, asked students to create examples of specific scenarios using the linear algebra they had learned during the semester. Two questions were selected for closer analysis: *Question 2: Construct a matrix A , not equal to the identity matrix, such that $A\mathbf{x} = \mathbf{b}$ is consistent for all \mathbf{b} . Be sure to justify why your example works.* *Question 4: Construct a non-standard basis of \mathbb{R}^3 and justify why your construction works.* These questions were selected for a closer look because they both asked students to perform at heightened levels of Bloom's Taxonomy and required understanding of different course concepts in the lower levels of Blooms Taxonomy, which are based in knowledge and comprehension. Question 2 is concerned with one fundamental idea of Linear Algebra, the idea of consistency, while question 4 is about two more advanced ideas, i.e., the concepts of a non-standard basis. While re-grading the exams, graders took note of the types of mistakes students were making frequently, focusing on students' ability to communicate their ideas. Graders then went back through each exam and tallied the number of students who made mistakes for each question. Table 4 shows the results.

Table 4. The most common mistakes made as noticed by the researchers, reported as a percentage of the number of students who took the exam (32 students in the flipped class, 34 in the traditional).

	Question 2		Question 4	
	Traditional	Flipped	Traditional	Flipped
1. Got fewer than 5/5 points on the question	81.1%	59.4%	83.8%	62.5%
2. Have completely wrong example or no answer	10.8%	6.3%	64.9%	40.6%
3. Making statements or claims without justifying/showing work	32.4%	12.5%	21.6%	9.4%
4. Have correct example without accurate reasons	70.3%	53.1%	21.6%	21.9%
5. Fail to explain ideas in readable sentence(s) (incl. no sentences)	43.2%	6.3%	70.3%	25.0%
6. Misuse of terminology/vague word choice in reasoning (excl. people who did not write any explanation)	70.3%	37.5%	37.8%	40.6%

The percentage of students who did not get full credit is comparable between classes, across questions. This shows that students in the same class (i.e., looking at only the flipped or only the traditional class) had the same level of ability to answer both the simpler question and the more complex one. This signifies that within the same class, overall student understanding of what was expected from construction questions was comparable regardless of the difficulty of the question. The heightened percentage in both classes of students with a completely wrong answer or no answer for question 4 shows that students in both classes are more likely to make a mistake when the problem involves multiple concepts, since the students not only need to fully grasp each concept individually, but be comfortable with the interaction between them.

When grading the selected questions closely, a repeated mistake that stood out was the difference in students' tendency to answer questions in a readable sentence. Readable sentences did not have to be grammatically correct or full sentences and they could include math symbols or operations. The only requirement was that the explanation could be read aloud in a reasonable way and make sense. The sentence would be counted as readable even if it was mathematically unsound or was defending an incorrect answer. The understanding that it is important to explain one's answer is an important concept in higher level mathematics, and the ability to do so exhibits an understanding by the

student of what he or she has done. Displaying a correct example of something with no explanation is akin to solving an equation for the unknown without showing any work. It means that the student can perform a task, but does not exhibit that they understood what they were doing or why it was correct. In both questions 2 and 4, students in the flipped class were much more likely to explain their ideas in a manner demonstrating that they had this understanding.

This tendency could be due to increased face-to-face attention with the instructor in the flipped class. Her constant interaction with and questioning of students, paired with the student interaction component of the group quizzes, got the students more accustomed to articulating why they knew their answer was right. They had to defend solutions to other students; everyone in the group shared a grade for each group quiz, so students had to agree on solutions before handing it in. Many groups formulated a quiz strategy where each student would tackle one or two problems. Every student wanted to be confident about the solutions that were handed in, so students got a lot of practice verbally defending their answers. This helped them gain familiarity with having to put their ideas in words, and gave them experiences much more like those they might face in a real-world work situation. In the traditional class, students did work in groups sometimes but for shorter periods and each student handed in his or her own work for a grade. The lack of group grade component meant that if students disagreed on an answer or the best way to do a problem, each could hand in their own paper and there was no impetus to explain their reasoning to peers.

Other possible explanations for the increased tendency of flipped students to explain their solutions in readable sentences are that they got more exposure to the type of question through the daily quizzes, had more graded feedback from the professor on this type of question when group quizzes were returned, and that the flipped instructor placed an emphasis in class on students ability to not only produce but justify correct solutions. Although any instructor would believe this is important, the use of class time for lecture in the traditional class did not allow for this important component to be brought home as effectively.

For the more basic question 2, students in the traditional class were almost twice as likely to misuse terminology or be vague in their explanation. These mistakes were grouped together because often students misused a word or relevant math phrase. On the other hand for more complex question 4, students in the flipped class had about the same frequency of misused terminology or unclear word choice as they did with question 2, while traditional students made this mistake less often than before, about as often as flipped students. This could signify that students in the flipped class retained relevant vocabulary and ideas more effectively over the course of the semester. They were able to explain the ideas behind a question on something they had learned at the beginning of the semester just as frequently as they could explain something from a much later, more recently learned chapter.

The traditional students, on the other hand, seemed to have trouble recalling less recently-learned information and how to use the relevant math words, but they were much more competent at the same task for a topic to which they had had more recent exposure. This difference could be due to flipped students gaining a higher-level understanding on Bloom's Taxonomy due to challenging group quizzes, and the need to explain and defend answers to peers during group quizzes for which one answer earned everyone the same grade. It could also be due to the in-class project done toward the end of the semester; if students needed to go back to less recent linear algebra topics for use in their real-life application example, they would have had a more recent exposure involving a deep use of their knowledge other than studying for the final exam.

As an example of our process in evaluating the quality of responses, Figures 1-5 include

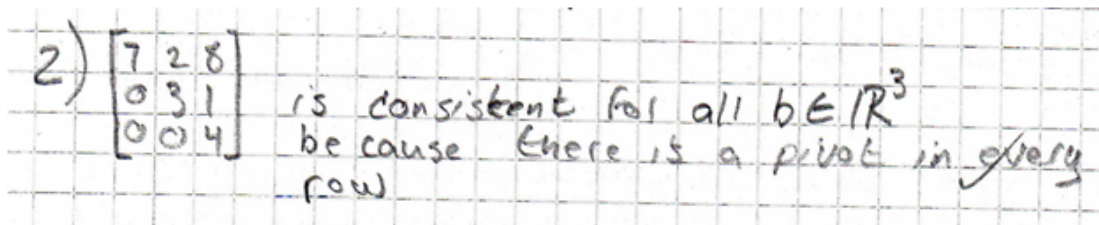


Figure 1. A “good” response to Question 2. Explanation reads, is consistent for all $b \in \mathbb{R}^3$ because there is a pivot in every row.

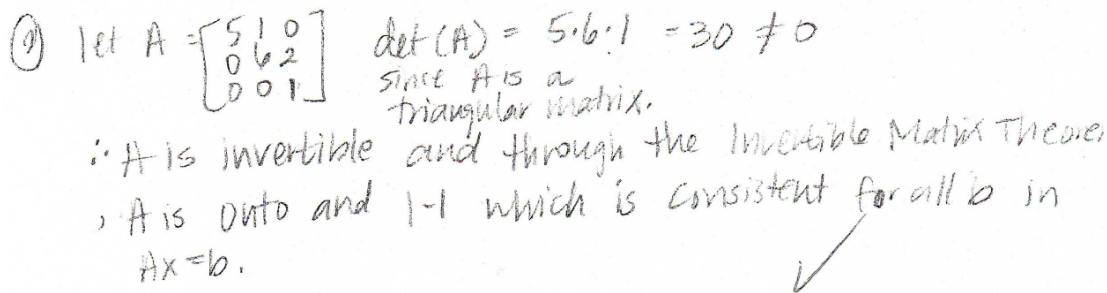


Figure 2. Another example of a “good” response. Explanation reads, “ $\det(A) = 5 \cdot 6 \cdot 1 = 30 \neq 0$ since A is a triangular matrix. $\therefore A$ is invertible and through the Invertible Matrix Theorem, A is onto and 1-1 which is consistent for all b in $Ax = b$.”

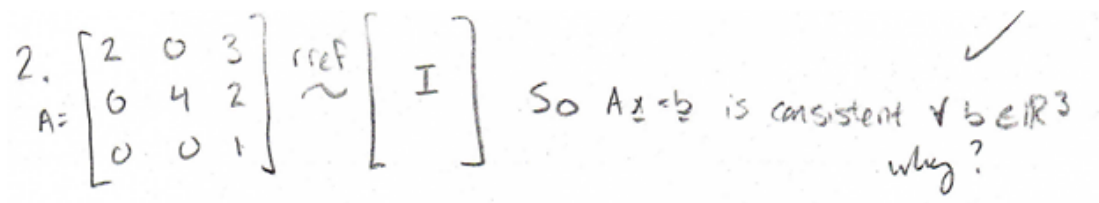


Figure 3. An example of a “medium” response. Explanation reads, “So $Ax = b$ is consistent $\forall b \in \mathbb{R}^3$,” with teacher comment, “why?”

a few selected responses from students in the flipped class to Question 2, “Construct a matrix A , not equal to the identity matrix, such that $Ax = b$ is consistent for all b . Be sure to justify why your example works.”

The first two responses, as illustrated in Figures 1 and 2, show what we considered a “good” answer. They gave an example that was correct, and explained how they knew in a complete and correct way, demonstrating excellent performance on a high level of Bloom’s Taxonomy. The responses in Figures 3 and 4 were considered “medium” responses. While the examples given are not technically incorrect, the explanations as to why the student knew their example fit the given criteria are incomplete and do not illustrate whether the student completely understood and was able to think on the higher level of Bloom’s Taxonomy, or simply got lucky. The final response, shown in Figure 5, is considered “poor.” The student gives an incorrect example. They showed a particular example of a vector that fits the requirements, instead of being able to show that it works for any vector. Thus this student does not have the necessary proficiency to receive credit on this problem because he or she does not understand the linear algebra or what it takes to prove something in mathematics.

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$? justification

Figure 4. Another example of a “medium” response. It shows two matrices and teacher comment, “? justification.”

2) $Ax = b$
 $\forall b$

$\begin{bmatrix} 1 & 2 & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

↑ when augmenting the matrix the rows show that the system is consistent and not the identity matrix. There can be infinitely many solu.

Figure 5. An example of a “poor” student response to Question 2. Explanation reads, “when augmenting the matrix the rows show that the system is consistent and not the identity matrix. There can be infinitely [sic] many solu.” Note that the $\forall b$ ” to the left side is a teacher comment.

3.3. Student Attitude Toward Mathematics Survey

A survey entitled “Mathematics Attitude Scale” was administered to both classes twice, in the first and last weeks of the semester. In the traditional class, 27 students took the pre-class survey and 32 students took the post-class survey; in the flipped class, 38 students took the pre-class survey and 33 took the post-class survey. The survey was administered through Google Forms and was adapted from the Fennema-Sherman Mathematics Attitude Scales and “Cooperative Learning in Calculus Reform: What Have We Learned?”. [9] It consisted of 45 statements in four categories: Confidence in Learning Mathematics (12 statements), Mathematics Usefulness (11 statements), Beliefs about Mathematics (11 statements), and Learning with Others (11 statements). Students chose their level of agreement with each statement on a scale of 1 (strongly disagree) to 5 (strongly agree) on a Likert Scale. In the first question, students were asked to provide the last four digits of their phone number and their mothers maiden name; this was in an effort to ensure that no repeat responses would be counted in the data while keeping the survey anonymous.

Responses were analyzed to detect differences over time and between classes, and the statements which showed significant differences in either of these analyses are given in Table 5, but the results and specifics behind the significance are discussed below. The full list of statements, including which category each belongs in, is given in Appendix B.

3.3.1. Pre/Post-Class Survey Comparison

To determine how student attitudes changed over time within the same class, we use the Wilcoxon Signed Rank Test, which is a nonparametric test for differences in the population median of paired samples of data. It is the nonparametric alternative to the paired samples t -test, and is used when the normality assumption cannot be justified. To determine which statements showed a significant difference from pre to post, and the direction of that difference, we use the following procedure: A p -value less than 0.05 indicates that there is a significant difference between pre- and post-agreement at the 5% significance level. To determine the sense of the inequality, the corresponding paired sample difference in mean value being positive indicates post > pre, meaning agreement increased over time; the same value being negative indicates that post < pre, meaning that agreement with the given statement decreased over time. This analysis is given in Tables 6 and 7.

Table 5. The statements for which there was a significant change in agreement over time or between classes.

Question Number	Category	Statement
2	Confidence in Learning Mathematics	I am sure I could do advanced work in mathematics.
8	Confidence in Learning Mathematics	I don't think I could do advanced mathematics.
11	Confidence in Learning Mathematics	Most subjects I can handle OK, but I have a knack for messing up in math.
12	Confidence in Learning Mathematics	Math has been my worst subject.
26	Beliefs about Mathematics	There are often several different ways to solve a math problem.
27	Beliefs about Mathematics	Time used to investigate why a solution to a math problem works is usually time well spent.
31	Beliefs about Mathematics	Math problems have one and only one right answer.
32	Beliefs about Mathematics	Math is mostly a matter of memorizing formulas and procedures.
38	Learning with Others	I prefer to work with other students when doing math assignments or studying for tests.
40	Learning with Others	Math is more interesting when I work in a group with other people.

Table 6. The median and mean scores of the significant items in the results of the Wilcoxon Signed Rank test on Attitude Survey responses from students in the **flipped classroom**, looking for changes in the same class over time.

Median(Mean)	Q2	Q8	Q26	Q27	Q31	Q40
Post	4(4.18)	2(1.79)	5(4.42)	5(4.39)	2(2.39)	4(3.79)
Pre	4(3.94)	1(1.81)	4(4.25)	5(4.19)	3(2.69)	4(3.77)
Paired Sample Diff. in Means	0.2326	-0.1395	0.0930	0.2093	-0.1628	0.2558
<i>p</i> -value	.038	.037	.035	.033	.013	.037

Table 7. The median and mean scores of the significant items in the Wilcoxon signed rank test on Attitude Survey responses from students in the **traditional classroom**, looking for changes in the same class over time.

Median(Mean)	Q11	Q12	Q38
Post	2(2.5)	1(1.35)	3(3.27)
Pre	1.5(1.85)	1(1.8)	2.5(3.12)
Paired Sample Diff. in Means	0.6842	0.5263	0.4737
<i>p</i> -value	.011	.008	.043

As shown in Table 6, there was a significant (since $0.038 < 0.05$) increase (since $0.2326 > 0$) in student perception of their ability to do advanced work in mathematics from before to after the flipped class (Q2). Also supporting this idea is the result from Q8, which shows that students agreed less with the statement “I don't think I could do advanced mathematics” at the end of their flipped experience than at the beginning. That is, perceived math abilities and confidence in math appears to have increased over time after learning in a flipped classroom.

The agreement changes by the flipped class on statements 26, 27, and 31 all show desirable changes in the category of beliefs about mathematics. Increased agreement with desirable statements 26 and 27 shows that students became more apt to be patient and open minded when working on math problems, likely because of the group interaction and forced cooperation; decreased agreement with undesirable statement 31 supports this sense of open-mindedness by showing that students lost faith in two commonly held, closed-minded math beliefs. The same test was used to look for changes in attitude in the traditional class over time; significant results are shown in Table 7.

While we saw a desirable change in flipped confidence and attitudes over time, we saw an undesirable change in the traditional class. Agreement with two undesirable statements, Q11 and Q12, increased (since both paired sample difference in means are positive) significantly (since both p -values are less than 0.05), showing that **students' traditional linear algebra experience made them feel worse about their math abilities**.

One desirable change that occurred was the significant increase in agreement by traditional students with Q38, "I prefer to work with other students when doing math assignments or studying for tests." Traditional students desire to work on math in a group changed desirably, though it is not evident from exam scores that group studying helped them in the way that it did in the flipped class. Additionally when we compare this to flipped students desirable attitude change in Q40, "Math is more interesting when I work in a group with other people," it seems possible that these are similar changes taking place in both classes, expressed by each class in a different way.

3.3.2. *Traditional/Flipped Comparison*

The second analysis performed on the Attitude Survey data was looking for significant differences between the two classes at either point in time. A Mann-Whitney U test was performed, and significant differences are discussed below. A key is used below in which a ** indicates that agreement for the flipped class was significantly greater; the absence of ** means that agreement was greater in the traditional class.

Some items were significantly different between the two groups in both pre- and post, indicating that the difference in opinion did not change over time, and was likely caused by something other than the class experience. However where we see the largest and most interesting difference in opinion between classes is in the category of beliefs about mathematics. Q29Post** (flipped > traditional) indicates that **students from the flipped class had a stronger belief that "the underlying mathematical ideas are more important than the formula" than the students in the traditional class** at the end of the semester without feeling this way in the beginning of the semester. This difference could be due to an increase in positive feeling towards the belief after experiencing the flipped learning or an increase in negative feeling towards the belief after experiencing the lecture-style learning.

On the other hand, the significance of Q32Post (traditional > flipped) indicates that students from the traditional class had a stronger belief that "math is mostly a matter of memorizing formulas and procedures" than the students in the flipped class at the end of the semester without feeling this way in the beginning of the semester. If we imagine the behavior of thinking "underlying ideas are more important than the formula" as a positive learning outcome and thinking "math is mostly a matter of memorizing formulas and procedures" as a negative learning outcome, then the coupled pair (Q29Post**, Q32Post) gives solid evidence that students in the flipped class finished the term with a stronger belief that the underlying mathematical ideas are more important than the formula. This is likely due to the format of the class.

3.4. *Flipped Perceptions Survey*

A second survey, "Perceptions of Student Learning in a Flipped Linear Algebra Class," was given only to students in the flipped class at the end of the semester. This survey was kept anonymous using the same method as the previously discussed survey. It was categorized into Effectiveness of Class Materials (questions 01-07), Study Habits (questions 08-13), Learning in the Flipped Classroom (questions 14-20), and Reflections (10 open-ended questions). The first three sections were quantitative and asked students to

choose their level of agreement with each statement on a Likert scale of 1 to 5. Thirty-two responses were recorded. Significant quantitative results are in Table 8; the statements not included were omitted because they were irrelevant to this study.

Table 8. A list of the statements from the Student Perceptions Survey which had meaningful responses, along with their scales and averages.

Question	Scale	Mean	Median
01. In general, I have no trouble following the flipped class schedule	1- Strongly Disagree; 5- Strongly Agree	4.59375	5
02. In general, I have no trouble accessing the online videos and finish watching the videos before class.	1- Strongly Disagree; 5- Strongly Agree	4.25	4.5
03. The exams were useful to my learning.	1- Strongly Disagree; 5- Strongly Agree	4.0833	4.333
03. The homework was useful to my learning.	1- Strongly Disagree; 5- Strongly Agree	4.0833	5
03. The in-class quizzes were useful to my learning.	1- Strongly Disagree; 5- Strongly Agree	4.45833	5
03. The online quizzes were useful to my learning	1- Strongly Disagree; 5- Strongly Agree	3.54167	3.667
03. The videos were useful to my learning.	1- Strongly Disagree; 5- Strongly Agree	4.20833	5
04. The due date and time for online quizzes are set reasonably.	1- Strongly Disagree; 5- Strongly Agree	4.46875	5
05. The in-class group quiz questions are thought-provoking and help me to deepen my knowledge.	1- Strongly Disagree; 5- Strongly Agree	4.5	5
06. The in-class group quiz questions allow me to gain confidence in my skill set.	1- Strongly Disagree; 5- Strongly Agree	4.0625	4.5
08. How many hours do you spend on out-of-class learning during a typical week (non-exam week)?	Self-Reported	7.03125	4.5
12. On average, how many hours do you spend studying for upcoming exams in this class?	Self-Reported	11.84375	6
13. On average, how many hours do you spend studying for upcoming exams in your previous math classes?	Self-Reported	8.484375	5
14. My in-class discussions with peers and the instructor help me learn.	1- Strongly Disagree; 5- Strongly Agree	4.28125	4.5
15. The class time is structured effectively for my learning.	1- Strongly Disagree; 5- Strongly Agree	4.03125	4
16. The class time is critical to my learning.	1- Strongly Disagree; 5- Strongly Agree	4.28125	5
17. The structure of this flipped class supports my learning in and out of class.	1- Strongly Disagree; 5- Strongly Agree	4.125	4
19. Having to communicate mathematics in class help me learn the concepts better.	1- Strongly Disagree; 5- Strongly Agree	4.34375	5
20. I enjoyed learning in this flipped class.	1- Strongly Disagree; 5- Strongly Agree	4.125	5

The responses to questions 01, 02, and 04 exhibit that in general, students had no trouble using the materials crucial to the flipped class and emphasize the importance of a strong structure to the class. Although a flipped format, in many ways, invites freedoms not available in other classes, the structure of the class is crucial. In this flipped class, the instructor had a website with the entire semester’s schedule posted, with links to all lecture videos, before the semester began. The ease of access and the early posting of the schedule allowed students to plan ahead more effectively, which is crucial for a flipped course as it demands more of the student outside of class than some other classes might. This is exhibited by the responses to questions 08, 12, and 13. Though we do not have a number of work hours to which we can compare a non-exam week in another math class, we can see that students felt like the time required to excel was elevated in this flipped class. The pre-planned and well-organized structure of the class allowed students to foresee exams, projects, and anything else.

They could, for example, watch and take notes on video lectures ahead of time so that all they had left to do before class was to take the LMS quiz, if they knew they had an event coming up that would decrease their available time outside of class. This also promoted organization for the students: they had to keep track of lectures, online and

in-class quizzes, and homework assignments. This helped students foresee the structure of the course and organize their work accordingly, **allowing them to be more efficient learners and succeed despite the additional time required.**

The difference in question 03 between the usefulness of the online and in class quizzes shows that students felt a significant difference as a result of collaboration. The online quizzes, administered through the university's Desire2Learn based Learning Management System, were taken independently. It was intended more to check comprehension and ensure timely viewing of the lectures, whereas the collaborative in-class quiz was much more challenging. As discussed above, it demanded teamwork and discussion among students to complete the difficult quiz in time. The in-class quizzes were also taken while the instructor moved around the classroom, listening and interjecting with critical questions. It is encouraging but not surprising that students saw the in-class quizzes as the most useful resource in the flipped classroom experience. **The flipped class is based on the idea that collaborative learning fosters deeper, more long-lasting learning, and the students clearly felt that this was true.**

The high averages for questions 05 and 06 are also encouraging. Although students struggled through the quizzes some days because of the designed difficulty level, they saw the value in their struggle. **Not only did students believe they learned more and were challenged during the group quizzes, their experiences with the quizzes helped them gain mathematical confidence.** This is supported in the free-response student questions, where 24 of the 32 students indicated that group quizzes and/or group work were the assignments from which they learned the most.

Possibly the most significant of the responses are the high averages in questions 14-20. These are related to the specific goals and intended benefits of the flipped classroom practice; they were outlined (though in different words) in the class syllabus and were emphasized by the instructor of the flipped class throughout the semester. The high averages in these questions show that students felt that the class time devoted to active problem solving was useful and helped them learn. Additionally, it shows that the goal of improved learning as a result of increased communication during class time seems to have been successful when measured with student perception, and seems to have been well-received by students, who reported enjoying the flipped experience.

When asked, 27 respondents thought that linear algebra was an appropriate class to be flipped; four students thought it was not an appropriate class to flip; and one responded, "I am not sure." Of the four who did not think that the class was appropriate to flip, three cited the difficulty of the course and the materials as their reason. Three of the four also responded that they were not getting the grade they wanted in the course (there were 17 students total that were not getting their desired grade). Twenty-one of 32 (65%) students said they would definitely be willing to take a flipped class again, and two others said they might, depending on the course. Ten students said that a main reason they would take another flipped class was because they learned more; others cited reasons like the videos, a support system, and enjoying their relationship with the instructor. Half of the eight students who would be unwilling to take another flipped class cited the additional time required to succeed as a reason, two did not give a reason, and two said they felt that they learned better in a traditional lecture-style setting.

When flipped students were asked to respond on whether they have noticed a change in the way they write (i.e., communicate) mathematics, two-thirds reported some change. One student responded, "Yes, proving a statement using numbers first is a good start into writing a formal proof thereafter," showing an understanding that a verbal explanation is needed to accompany a mathematical one. Another student wrote, "Yes, I learned how to formally and specifically express mathematical concepts in a detailed manner ... I now define concepts as explicitly as possible and in the correct scope that it is used in. Also, I

now know how to answer true and false questions by giving sufficient counterexamples.” As discussed in earlier sections, math communication like this was a desired outcome in the flipped class and was demonstrated by student responses, but we see through this response and others that students felt the change in behavior as it was being developed. A complete data set of student responses on this and the attitude survey is available to interested readers who contact the corresponding author.

4. Summary / Future Work / Recommendations

This flipped class case study suggests that in this Introductory Linear Algebra class, deeper learning was exhibited through test scores and examination of open-ended student responses to critical questions. It uses quantitative data about student performance and perceptions which has not been seen in existing literature to show the effectiveness of an appropriately structured flipped class. Our data and analyses showed improved student beliefs, confidence, and attitudes about mathematics, supporting research such as Love et al. by seeing that flipped students believed more that linear algebra was important to their careers.[5] Students in this flipped class also gained valuable communication skills and practice communicating mathematical ideas, and clearly felt that their improved learning was worth the extra work it took. As a consequence, we believe that when a class is organized effectively and efficiently, deeper learning can be accomplished through various means of active learning strategies even in a lecture-based or hybrid class.

While this study provides encouraging news to educators who wish to infuse their classes with more active learning activities, we urgently need more educational research to examine different modes of instruction, the effects these instructional methods have on students, and the contexts in which the instruction takes place. A focus should remain on student performance and success in subsequent mathematics courses, to either further support or show evidence against claims of improved learning results from students in flipped classrooms.

To those considering a flipped class, we recommend doing so gradually. For example, begin by replacing one regular lecture with a flipped lesson. This allows instructors to work out kinks and modify the course based on student reactions. When running a flipped class, there is not a one-size-fits-all bucket list of must-dos. Do what works for your teaching style and your institution, as suggested by [10–12]. We emphasize the importance of a strong structure. It has been noted in previous research by Strayer, and was reaffirmed in this study that students can often be resistant to the change to their work and study habits brought by the flipped format. Increased predictability and organization can help ease the transition and empower students to find a rhythm that works for them.[1] Accessibility to online materials at all times and approachability of the instructor also rank high on the list of best practices. It is imperative that the instructor maintain a manageable workload for students. For example, post-class homework proved to be emotionally taxing and time-consuming for students who have mastered the materials from the pre-class and in-class assignments. It might not be a bad idea to make it optional or even given as extra credit at the discretion of the instructor. Finally, instructors should be aware of how individuals are learning and interacting with the group to provide just-in-time interventions when necessary. It is the student-student and student-faculty interactions that contribute greatly to the success of the flipped learning.

There are definitely psychological barriers when one decides to deviate from a lecture-based teaching paradigm, including fear of uncertainty and possible misconceptions of how classroom management might affect student learning. Having reviewed the comparative results, the instructor of the traditional class is definitely open to the possibility of teaching linear algebra in a completely flipped format. Both instructors felt that a strong

familiarity of the learner's background and the course content are prerequisite to a successful implementation of flipped classes. Designing a full-blown flipped class requires a lot more preparation time, sometimes three times more than designing a lecture-based class; therefore an interested instructor should be cautious and gauge their familiarity with the technology, mastery of the content, the expected student learning outcomes, and the program learning outcomes before embarking on this endeavor. Fortunately, students as well as teachers of a flipped classroom can benefit from having gone through a full iteration once. For teachers, the idea of a backward-design process to first detail expected student learning outcomes then create materials to facilitate learning and measure success is a concept that is transferrable to any class. For students, a highly predictable schedule in a flipped class forces them to manage time wisely and take ownership of their own learning. Both skills help making them successful students elsewhere. In conclusion, the results of this study indicate that increased student engagement with advanced mathematics content in a supervised setting may yield benefits that are worth the risk.

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Appendix A. Common Final Exam

This section contains the final exam that was given to the flipped class as well as the traditional class.

Part 1: Definitions/Fill In The Blank

- (1) (5 points) A set $\{v_1, v_2, \dots, v_p\}$ is **linearly independent** if _____.
- (2) (5 points) An $m \times n$ matrix U is **orthogonal** if _____.

Part 2: Computations

- (1) (5 points each) Given $A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$.
- Find all eigenvalues of A . Be sure to justify your reasonings.
 - Find all eigenvalues of A^5 . Be sure to justify how you found them.
 - Find a basis for each eigenspace of A .
 - Is A diagonalizable? Why or why not?
 - Is A^5 invertible? Why or why not?
- (2) (40 points) Given

$$\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Let $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$ (matrix) and $S = \{\mathbf{a}_1, \mathbf{a}_2\}$ (set).

- (5 points) Determine whether S is an **orthogonal** basis for $\text{Col } A$. Justify your answer.
- (10 points) Apply the Gram-Schmidt Process to find an **orthonormal** basis for $\text{Col } A$.
- (4 points) Find a nonzero vector in \mathbb{R}^3 that is orthogonal to $\text{Col } A$. Be sure to justify your work.
- (5 points) Find the QR -factorization of A .
- (6 points) Is the system $A\mathbf{x} = \mathbf{b}$ consistent? Justify your answer.
- (10 points) If your answer to part (e) is yes, solve for the solutions, \mathbf{x} . If your answer to part (e) is no, find the least-squares solution to the system $A\mathbf{x} = \mathbf{b}$ using any method of your choice.

Part 3: True/False in Multiple Choice (Note: there is only one answer to every question below)

- (5 points) Let A and B be some general $n \times n$ matrices. Which of the following is **true** for A and B ?
 - An elementary row operation on A does not change the determinant of A .
 - $\det(AB) = \det(BA)$.
 - $\det(A + B) = \det A + \det B$.
- (5 points) Which of the following is **false**?
 - In some cases, a plane in \mathbb{R}^3 is isomorphic to \mathbb{R}^2 .
 - If \mathcal{B} is the standard basis for \mathbb{R}^n , then \mathcal{B} -coordinate vector of an \mathbf{x} in \mathbb{R}^n is \mathbf{x} itself.
 - If $P_{\mathcal{B}}$ is the change-of-coordinate matrix from \mathcal{B} to the standard basis in \mathbb{R}^n , then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$ for all \mathbf{x} in a vector space V .

- (3) (5 points) Which of the following is **true**?
- (a) For any $m \times n$ matrix A , $\dim \text{Col } A + \dim \text{Nul } A^T = m$.
 - (b) If $\dim V = n$ for a vector space V and S is a linearly independent set in V , then S is a basis for V .
 - (c) If B is any echelon form of A , then the pivot columns of B form a basis for the column space of A .
- (4) (5 points) Which of the following is **true**?
- (a) If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ span a subspace W and if \mathbf{x} is orthogonal to each \mathbf{v}_j for $j = 1, \dots, p$, then \mathbf{x} is in W^\perp .
 - (b) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues counting multiplicities.
 - (c) If a matrix A is invertible, then A is diagonalizable.
- (5) (5 points) Which of the following is **false**?
- (a) A square matrix A is invertible if and only if A has nonzero determinant.
 - (b) A square matrix A is invertible if and only if A has orthogonal columns.
 - (c) A square matrix A is invertible if and only if A does not have zero eigenvalues.
- (6) (5 points) Which of the following is **true**?
- (a) A plane in \mathbb{R}^3 is a two-dimensional subspace.
 - (b) If A is $m \times n$ and $\text{rank } A = m$, then the linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
 - (c) If A is $m \times n$ and the linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is onto, then $\text{rank } A = m$.

Part 4: Application

- (1) (20 points) A simple curve that makes a good model for the variable costs of a company, as a function of the sales level x , has the form $y = \beta_0 + \beta_1 x + \beta_2 x^2$. Find the **least-squares** curve of the form above to fit the data $(4, 1.58)$, $(6, 2.08)$, $(8, 2.5)$, $(10, 2.8)$, with values in thousands. That is, find β_0 , β_1 , and β_2 in the equation. Round your answers to the first 4 decimal digits if necessary.

Part 5: Construction of Examples

- (1) Given that \mathbb{R}^3 is a vector space.
- (a) (5 points) Construct a 2-dimensional subspace of \mathbb{R}^3 and justify why your example works.
 - (b) (5 points) Construct a subset of \mathbb{R}^3 that is **not** a subspace of \mathbb{R}^3 and justify why your example works.
- (2) (5 points) Construct a matrix A , not equal to the identity matrix, such that $A\mathbf{x} = \mathbf{b}$ is consistent for all \mathbf{b} . Be sure to justify why your example works.
- (3) (5 points) Construct a matrix A such that its associated homogeneous system has nontrivial solutions. Be sure to justify why your example works.
- (4) (5 points) Construct a non-standard basis of \mathbb{R}^3 and justify why your construction works.
- (5) (Optional: Extra Credit 5 points) Construct an orthogonal projection matrix \mathbb{P} that will take any point in \mathbb{R}^3 to span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$. (Hint: utilize the work you have already done in **Part 2 #2**.)

Appendix B. Mathematics Attitude Scale Survey

The “Mathematics Attitude Scale” survey administered to both classes in the beginning and at the end of the semester.

Confidence in Learning Mathematics The Mathematics Usefulness Scale is designed to measure confidence in one’s ability to learn and perform well on mathematical tasks. (Fennema & Sherman, 1986, page 4).

- 01 Generally I feel secure about attempting to learn mathematics.
- 02 I am sure I could do advanced work in mathematics.
- 03 I am sure that I can learn mathematics.
- 04 I can get good grades in mathematics.
- 05 I have a lot of self-confidence when it comes to math.
- 06 I think I could handle more difficult mathematics.
- 07 I am not good in math.
- 08 I don’t think I could do advanced mathematics.
- 09 I am not the type to do well in math.
- 10 For some reason even though I study, math seems unusually hard for me.
- 11 Most subjects I can handle OK, but I have a knack for messing up in math.
- 12 Math has been my worst subject.

Mathematics Usefulness The Mathematics Usefulness Scale is designed to measure students’ beliefs about the usefulness of mathematics currently or in relationship to their future education, vocation, or other activities (Fennema & Sherman, 1986, page 5).

- 13 I will need mathematics for my future work.
- 14 I study mathematics because I know how useful it is.
- 15 Knowing mathematics will help me earn a living.
- 16 Mathematics is a worthwhile and necessary subject.
- 17 I will need a firm mastery of mathematics for my future work.
- 18 I will use mathematics in many ways in my life.
- 19 Mathematics has no relevance to my life.
- 20 Mathematics will not be important to me in my life’s work.
- 21 I see mathematics as a subject I will rarely use in my daily life after college.
- 22 Taking mathematics is a waste of time.
- 23 In terms of my adult life, it is not important for me to do well in mathematics in college.

Beliefs About Mathematics Beliefs About Mathematics Scale is intended to measure the degree to which students view mathematics as a creative, thoughtful enterprise, as opposed to a static body of algorithmic knowledge (Herzig & Kung, 2003, Page 25).

- 24 In math, you can be creative and discover things by yourself.
- 25 The math I learn in school is thought-provoking.
- 26 There are often several different ways to solve a math problem.
- 27 Time used to investigate why a solution to a math problem works is usually time well spent.
- 28 In addition to getting a right answer in mathematics, it is important to understand why the answer is correct.
- 29 The underlying mathematical ideas are more important than the formulas.
- 30 Just about everything important about math is already known by mathematicians.
- 31 Math problems have one and only one right answer.
- 32 Math is mostly a matter of memorizing formulas and procedures.
- 33 To solve math problems, you have to know the exact procedure for each problem.

34 Students who understand the math they have studied will be able to solve any assigned problem in five minutes or less.

Learning With Others The Learning With Others Scale is intended to measure the degree to which students believe that working with others is helpful in learning mathematics (Herzig & Kung, 2003, Page 25).

35 When I can't understand material in a math class, I like to ask another student in class for help.

36 Studying math with others helps me see different ways to solve problems.

37 Talking with others about math problems helps me understand better.

38 I prefer to work with other students when doing math assignments or studying for tests.

39 I work harder when I work in a group with other students.

40 Math is more interesting when I work in a group with other people.

41 When I become confused about something I am studying in math, I go back and try to figure it out myself.

42 When study math with other students, we don't get much done.

43 I learn math best when I study by myself.

44 When I work on math with other students, I usually end up doing more than my share of the work.

45 It is hard to work with other students on math because some students work faster or slower than others.