Topology Comprehensive Exam

Name:____________________

Answer six (6) questions total. On the first page of your work, please write the numbers of the problems that you want graded. On each page please write only on the front side.

Problems							Total
Scores							

1.

(a) Define what it means for a topological space to be Hausdorff.

(b) Define the subspace topology.

(c) Show that every infinite Hausdorff space has an infinite discrete subspace. [Hint: Separate into two cases: Case 1. Every point p in X is discrete, i.e. p has a neighborhood that contains p alone. Case 2. X has a point p, so that every neighborhood of p contains at least one point other than p.]

2. For both of the following determine whether the topological space X must be compact, cannot be compact, or could be either. In each case, provide proof (if it could be either, provide examples with explanation).

(a) X is a topological space with an infinite subset Y such that every subset of Y is closed.

(b) X is a topological space whose only infinite closed set is X

3. Let $\mathbb{R}_K$ be the set of real numbers with the topology with basis $\{(a,b) \mid a,b \in \mathbb{R}, a > b\} \cup \{(a,b) - K \mid a,b \in \mathbb{R}, a > b\}$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$.

(a) Determine whether $\mathbb{R}_K$ is connected.

(b) Determine whether $\mathbb{R}_K$ is path connected.

4.

(a) Define what it means for a topological space to satisfy the 1st countability axiom

(b) Define what it means for a topological space to satisfy the 2nd countability axiom

(c) Prove that if X satisfies the 1st countability axiom and $f : X \to Y$ is a continuous open map, then the image of X under f also satisfies the 1st countability axiom.

(d) Prove that if X satisfies the 2nd countability axiom and $f : X \to Y$ is a continuous open map, then the image of X under f also satisfies the 2nd countability axiom.

5. Given a subset S in a topological space X, let $\overline{S}$ be its closure, and S° be its interior.

For the following statements about subsets A, B in X, prove it if it is true, provide a counterexample if it is not true.

(a) $A^\circ \cup B^\circ = (A \cup B)^\circ$.

(b) $A^\circ \cap B^\circ = (A \cap B)^\circ$.

(c) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(d) $\overline{A} \cap \overline{B} = \overline{A \cap B}$.

6.

(a) Let X be a topological space such that for every topological space P and any map $f : X \to P$, f is continuous. Prove that X has the discrete topology.

(b) Let Y be a topological space such that for every topological space Q and any map $g : Q \to Y$, g is continuous. Prove that the only open sets in Y are $\emptyset$ and Y.

7.

(a) Assume $f : X \to Y$ is continuous. Assume $x_n \to x$. Prove that $f(x_n) \to f(x)$.

(b) Assume $f : X \to Y$ is a map, and X is first countable. Assume for any convergent sequence x_n in X, $f(x_n) \to f(\lim x_n)$. Prove that f is continuous.

8. Let X be the set of sequences of real numbers $(x_1, x_2, x_3, ...)$ satisfying that there are only finitely many nonzero x_n's. Therefore, X is a subspace of $\mathbb{R} \times \mathbb{R} \times ...$ with the product topology; let τ_1 be the subspace topology on X.

Let τ_2 be the topology on X defined by the distance function

$$d\big((x_1, x_2, ...), (y_1, y_2, ...)\big) = \sum_{k=1}^{\infty} |x_k - y_k|.$$

(a) Prove that every open set in τ_1 is also open in τ_2.

(b) Show that there exist open sets in τ_2 that are not open in τ_1.

9. Let X be the topological space of $\mathbb{R}^2$ with the order topology coming from the lexicographical ordering (also known as the dictionary order topology).

(a) Define a quotient map from X to $\mathbb{R}$ with the standard topology. Prove your map is a quotient map.

(b) Define a quotient map from X to $\mathbb{R}$ with the discrete topology. Prove your map is a quotient map.

10. Let $\mathbb{R}_K$ be the topological space defined in problem **3.**.

(a) Prove that $\mathbb{R}_K$ is Hausdorff,.

(b) Prove that $\mathbb{R}_K$ is not T_3.