CSULB Topology Comprehensive Exam, Feb. 15, 2025.

Answer six (6) questions total. On each page of your answers, please do not write anything on the back of that page.

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1). Determine whether the following statements about *closure* are true. If a statement is true, please give a proof. If a statement is not true, give a counter example.

i. $\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup ... = \overline{A_1 \cup A_2 \cup A_3 ...}$

ii. $\overline{A - B} = \overline{A} - \overline{B}$.

iii. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

2). i. Assume $X = \bigcup_{k=1}^{\infty} W_k$, where W_k is open for any k. Assume $f : X \to Y$ is a map, so that $f|_{W_k}$ (i.e. the restriction of f on W_k, viewed as a map $W_k \to Y$) is continuous for any k. Prove that f is continuous.

ii. Assume $X = A \cup B$, where A, B are closed. Assume $f : X \to Y$ is a map, so that $f|_A$ and $f|_B$ are continuous. Prove that f is continuous.

iii. Assume $X = \bigcup_{k=1}^{\infty} E_k$, where E_k is closed for any k. Assume $f : X \to Y$ is a map, so that $f|_{E_k}$ is continuous for any k. Is it always true that f is continuous? (Prove it or give a counterexample.)

3). i. Prove that if X is connected, $f : X \to Y$ is continuous, then $f(X)$ is connected.

ii. If the product space $\prod_{\alpha \in A} X_\alpha$ contains a nonempty connected open set U, prove that X_α is connected for all but finitely many α.

4). Let τ be the smallest topology on $\mathbb{R}^2$ such that the intersection of any two lines is open.

i. Is $(\mathbb{R}^2, \tau)$ 1st countable?

ii. Is $(\mathbb{R}^2, \tau)$ 2nd countable?

iii. Is $(\mathbb{R}^2, \tau)$ metrizable?

Give your reasons.

5). Define an equivalence relation $\sim$ on $\mathbb{R}^2$: $(x, y) \sim (a, b)$ if and only if

$$3x - 5y = 3a - 5b.$$

Prove that this is an equivalence relation, and the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

6). i. Assume X is a metric space with distance d. For a subset $S \subset X$, define the function d_S by

$$d_S(x) = \inf_{q \in S} d(x, q).$$

Prove that f is continuous.

ii. Prove that any metric space is normal.

7). Assume $X_1, X_2, X_3, ...$ is a sequence of topological spaces, $Y_1, Y_2, Y_3, ...$ is a sequence of topological spaces. Assume for each $j = 1, 2, 3, ...$, there is a map $f_j : X_j \to Y_j$. Define

$$f : \prod_{j=1}^{\infty} X_j \longrightarrow \prod_{j=1}^{\infty} Y_j$$

where both domain and range have the product topology, by

$$f(x_1, x_2, x_3, ...) = \big(f_1(x_1), f_2(x_2), f_3(x_3), ...\big).$$

Prove that f is continuous if and only if every f_j is continuous.

8). i. Give an example of a topological space X, and a sequence of nonempty closed subsets $A_1, A_2, ... \subset X$, so that $A_1 \supset A_2 \supset A_3 \supset ...$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

ii. Assume X is compact, $A_1 \supset A_2 \supset A_3 \supset ...$ is a sequence of closed nonempty subsets. Prove that $\bigcap_{n=1}^{\infty} A_n$ is not empty.

9). i. Prove that a closed subset of a compact space is compact.

ii. Prove that in a Hausdorff space, for any compact set K and a point $p \notin K$, there are disjoint open sets U, V such that $K \subset U$ and $p \subset V$.

10). Consider the product space $X = \prod_{n=1}^{\infty} [0, 1]$, with the product topology.

i. Prove that X is Hausdorff.

ii. Prove that X is separable.