Topology Comprehensive Exam

Name:____________________

Answer six (6) questions total. On the first page of your work, please write the numbers of the problems that you want graded. On each page please write only on the front side.

Problems							Total
Scores							

1. Given any set $X \neq \emptyset$, prove that there is a smallest topology on X such that X is T_1. What is this topology? Prove that your answer works, that is, that it is a topology, that it T_1, and that it is the smallest T_1 topology on X.

2. Let (X, τ) be a topological space. Let $\mathcal{C} \subseteq \tau$ be a collection of sets with the property that given any $U \in \tau$ and any $x \in U$, there exists $C \in \mathcal{C}$, such that $x \in C \subseteq U$.

a. Prove that $\mathcal{C}$ is a basis.

b. Prove that $\mathcal{C}$ is a basis for the topology τ.

3. Let (X, τ) be a topological space, and let $\mathcal{C}$ be the collection of closed sets. A *filter* on $\mathcal{C}$ is a collection $\mathcal{F}$ of sets from $\mathcal{C}$, such that (1) $\emptyset \notin \mathcal{F}$, (2) if $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{F}$, and (3) if $C_1 \subset C_2$ with $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{C}$, then $C_2 \in \mathcal{F}$. Show that if each filter on $\mathcal{C}$ has non-empty intersection, then (X, τ) is compact.

4. Let $f : \mathbb{R} \to \prod_{i \in \mathbb{Z}^+} \mathbb{R}$ be the map given by $f(t) = (t, \frac{1}{2}t, \frac{1}{4}t, \frac{1}{8}t, \frac{1}{16}t...)$.

a. Define the box topology on $\prod_{i \in \mathbb{Z}^+} \mathbb{R}$.

b. Prove or disprove that f is continuous when $\prod_{i \in \mathbb{Z}^+} \mathbb{R}$ has the **box topology** and $\mathbb{R}$ has the standard topology.

c. Let $d : \mathbb{R}^2 \to \mathbb{R}$ be defined by $d(x, y) = \min\{|x - y|, 1\}$. Given points $\vec{x}, \vec{y} \in \prod_{i \in \mathbb{Z}^+} \mathbb{R}$, let $\rho(\vec{x}, \vec{y}) = \sup\{d(x_i, y_i) \mid i \in \mathbb{Z}^+\}$. You may assume that d and ρ are metrics. The metric topology for ρ is called the *uniform topology* on $\prod_{i \in \mathbb{Z}^+} \mathbb{R}$. Prove or disprove that f is continuous when $\prod_{i \in \mathbb{Z}^+} \mathbb{R}$ has the **uniform topology** and $\mathbb{R}$ has the standard topology.

5. Consider the product topology $X = \prod_{\alpha \in \mathbb{R}} \mathbb{R}_\alpha$, where each $\mathbb{R}_\alpha$ is a copy of the real line $\mathbb{R}$ using the standard topoolgy, i.e. X is "the product of $\mathbb{R}$ many real lines". A point $(x_\alpha)_{\alpha \in \mathbb{R}}$ [i.e. the α-component of this point is $x_\alpha \in \mathbb{R}$] can be identified with a function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(\alpha) = x_\alpha$.

Let f_i be a sequence in X. Show that $f_i \to f$ if and only if, viewed as functions, f_i converges to f pointwise [i.e. for any $x \in \mathbb{R}$, we have $\lim_{i \to \infty} f_i(x) = f(x)$.]

6. For each of the following equivalence relations on topological spaces, give the familiar space to which the quotient space $X/\sim$ is homeomorphic. You DO NOT need to justify your answer.

a. Define an equivalence relation on $X = \mathbb{R}$ with the standard topology by setting
$$x_1 \sim x_2 \quad \text{if} \quad x_1 - x_2 \in \mathbb{Z}.$$

b. Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting
$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad x_1 - x_2 \in \mathbb{Z} \quad \text{and} \quad y_1 - y_2 \in \mathbb{Z}.$$

c. Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting
$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad y_1 - (x_1)^3 = y_2 - (x_2)^3.$$

d. Define an equivalence relation on $X = \mathbb{R}^2$ with the standard product topology by setting
$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad (x_1)^2 + (y_1)^2 = (x_2)^2 + (y_2)^2.$$

e. Let X be the topological space $\mathbb{R}^2$ with the dictionary order topology. Define an equivalence relation on X by setting
$$x_1 \times y_1 \sim x_2 \times y_2 \quad \text{if} \quad x_1 = x_2.$$

7. Prove the following (you may use standard results so long as they do not trivialize the problem):

a. Assume X, Y are path-connected, then $X \times Y$ is path-connected.

b. Assume X, Y are connected, then $X \times Y$ is connected.

8. Assume X is compact, Hausdorff. Assume there is a sequence of compact subsets $A_1 \supset A_2 \supset A_3 ...$ in X. Prove the following (you may use standard results so long as they do not trivialize the problem):

a. If for all n, $\bigcap_{j=1}^{n} A_j \neq \emptyset$, prove that $\bigcap_{j=1}^{\infty} A_j \neq \emptyset$.

b. Assume U is an open set so that $\bigcap_{j=1}^{\infty} A_j \subset U$. Prove that for sufficiently large n, $A_n \subset U$. [hint: find an open cover of $X - U$.]

9. A topological space X is called *Lindelöf* if every open cover $\mathcal{U}$ of X has a countable subset $\mathcal{V} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup_{V \in \mathcal{V}} V$. A topological space X is called *Lindelite* if every open cover $\mathcal{U}$ of X has a countable subset $\mathcal{V} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup_{V \in \mathcal{V}} \overline{V}$.

a. Define T_3 (also known as "regular").

b. Suppose that X is T_3. Prove that if $x \in U$ where U is open, then there exists an open set W such that $x \in W \subset \overline{W} \subset U$.

c. Prove that if X is Lindelite and T_3, then X is Lindelöf.

10. Let, $X = \{1, 2, 3, 4\}$ and $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Note that (X, τ) is a topological space.

a. Define what it means for a topological space to be normal.

b. Prove that (X, τ) is normal.

c. Prove that $A = \{1, 2, 3\} \subset X$ with the subspace topology is not normal.