

Real Analysis Comprehensive Examination

Feb. 2026

Choose six of the nine problems, and mark the problems that you want graded on the table below.

Question	To Be Graded (✓)
1	
2	
3	
4	
5	
6	
7	
8	
9	

Notation:

- If $A \subseteq \mathbb{R}$, $|A|$ denotes the Lebesgue Outer Measure of A .
- $\mathcal{B}$ denotes the Borel measurable subsets of $\mathbb{R}$
- λ denotes the Lebesgue Measure on the Real line
- $(X, \mathcal{S}, \mu)$ will denote a measure space
- $\mathcal{L}^1(X, \mathcal{S}, \mu) := \{f \text{ measurable} : \int_X |f| d\mu < \infty\}$
- $\|f\|_1 := \int_X |f| d\mu$

1. Let $E \subseteq \mathbb{R}$ be measurable with $\lambda(E) < \infty$.

(a) Let $\{t_n\}_{n=1}^\infty$ be a sequence of real number with $\lim_{n \rightarrow \infty} t_n = 0$. Show that

$$\lim_{n \rightarrow \infty} \lambda(E \cap (E + t_n)) = 0$$

(b) Conclude that that

$$\lim_{t \rightarrow 0} \lambda(E \Delta (E + t)) = 0.$$

2. Let $t \in \mathbb{R}$. Show that for any Lebesgue measurable set, we have that $E + t$ is also Lebesgue measurable.

3. Prove or give a counterexample: if $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, \lambda)$, then $\lim_{x \rightarrow \infty} f(x) = 0$.

4. Let $f : X \rightarrow \mathbb{R}$ be measurable and bounded. Show that for each $\varepsilon > 0$. there exists a bounded measurable step function $g : X \rightarrow \mathbb{R}$ so that for all $x \in X$,

$$|f(x) - g(x)| < \varepsilon$$

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that f is absolutely continuous, $f' \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, \lambda)$. Show that $\lim_{x \rightarrow \infty} f(x)$ exists.

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that for all $c \in \mathbb{R}$ and all $r > 0$, we have $\int_{c-r}^{c+r} f(x) d\lambda(x) \geq 2r$. Show $f(x) \geq 1$ almost everywhere.

7. Suppose that $f_n, f \in \mathcal{L}^1([0, 1], \mathcal{B}, \lambda)$, $f_n \rightarrow f$ pointwise on $[0, 1]$. Does it follow that $\int_{[0, 1]} f_n d\lambda \rightarrow \int_{[0, 1]} f d\lambda$.

8. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be the following function:

$$f_n(x) = nx\chi_{[0, 1/n]}(x)$$

Does f_n converge to the zero function pointwise? what about in $\|\cdot\|_1$?

9. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be given as follows:

- $f_1(x) = 1$;
- $f_2(x) = 1$ if $x \in [0, 1/2]$, zero otherwise;
- $f_3(x) = 1$ if $x \in [1/2, 1]$, zero otherwise;
- $f_4(x) = 1$ if $x \in [0, 1/3]$, zero otherwise;
- $f_5(x) = 1$ if $x \in [1/3, 2/3]$, zero otherwise;
- $f_6(x) = 1$ if $x \in [2/3, 1]$, zero otherwise;
- $f_7(x) = 1$ if $x \in [0, 1/4]$, zero otherwise;

(a) Does f_n converge to the zero function pointwise?

(b) What is $\lim_{n \rightarrow \infty} \|f_n\|_1$?

10. Suppose $(X, \mathcal{S}, \mu)$ is a measure space $f : X \rightarrow \mathbb{R}$ is measurable, and assume that $\int_X |f|^p d\mu < \infty$. Show that for any $k > 0$ we have

$$\mu(\{x : |f(x)| \geq k\}) \leq \frac{1}{k^p} \int_X |f|^p dx$$

11. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable with $0 < \lambda(E) < \infty$. Set $f : \mathbb{R} \rightarrow \mathbb{R}$ to $f(x) = \lambda((-\infty, x) \cap E)$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Show that for any $b \in (0, \lambda(E))$, there exists some $x \in \mathbb{R}$ so that $f(x) = b$.

12. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f_n, f : X \rightarrow \mathbb{R}$ measurable.

(a) State Fatou's Lemma.

(b) State the Dominated Convergence Theorem.

(c) Derive the Dominated Convergence Theorem as a consequence of Fatou's lemma.

13. Compute

$$\lim_{n \rightarrow \infty} \int_0^1 \left[1 - (-1)^n e^{-x^2/n} \right] x^{-1/2} d\lambda(x)$$

14. Let $(X, \mathcal{S}, \mu)$ be a measure space.

(a) State Egoroff's theorem.

(b) Alternate question: Let $f_n, f : X \rightarrow \mathbb{R}$ be measurable, $f_n \rightarrow f$ pointwise, and $X = \bigcup_{k=1}^{\infty} X_k$ where $\mu(X_k) < \infty$ for all $k \in \mathbb{N}$. Show that for all $\varepsilon > 0$, there exists a set $E \in \mathcal{S}$ so that

- $\mu(X \setminus E) < \varepsilon$, and
- $f_n \rightarrow f$ uniformly on E .