

Real Analysis Comprehensive Examination

Sep. 2025

Choose six of the nine problems, and mark the problems that you want graded on the table below.

Question	To Be Graded (✓)
1	
2	
3	
4	
5	
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9	

Notation:

- If $A \subseteq \mathbb{R}$, $|A|$ denotes the Lebesgue Outer Measure of A .
- $\mathcal{B}$ denotes the Borel measurable subsets of $\mathbb{R}$
- λ denotes the Lebesgue Measure on the Real line
- $(X, \mathcal{S}, \mu)$ will denote a measure space
- $\mathcal{L}^1(X, \mathcal{S}, \mu) := \{f \text{ measurable} : \int_X |f| d\mu < \infty\}$
- $\|f\|_1 := \int_X |f| d\mu$

1. If $E \subseteq \mathbb{R}$, we define $|E|_* := \sup \{|A| : A \subseteq E, \text{ and } A \text{ is closed and bounded}\}$.
 - (a) Show that $|E|_* \leq |E|$ for all $E \subseteq \mathbb{R}$.
 - (b) Show that if E is bounded and E is Lebesgue measurable, then $|E|_* = |E|$.
 - (c) Show that if $|E| < \infty$ and E is Lebesgue measurable, then $|E|_* = |E|$. Hint: Notice

$$|E| = \sum_{n=0}^{\infty} |E \cap ([-n-1, -n) \cup [n, n+1))|.$$

- (d) Show that if $|E| = \infty$ and E is Lebesgue measurable, then $|E|_* = |E|$. Hint: a finite union of closed sets is closed.

2. Let $(X, \mathcal{S}, \mu)$ be a measure space.

- (a) Show that if $A \cap B = \emptyset$ and $A, B \in \mathcal{S}$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.
- (b) Show that if $A, B \in \mathcal{S}$, $A \subseteq B$, and $\mu(B) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.
Briefly explain why the hypothesis that $\mu(B) < \infty$ required here.
- (c) Let $\{A_i\}_{i=1}^{\infty}$ be sets in $\mathcal{S}$ with $\mu(\cup_{i=1}^{\infty} A_i) < \infty$. Show that

$$\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^j A_i\right) \geq \limsup \mu(A_i).$$

3. For $f \in \mathcal{L}^1([0, 1], \lambda)$, we define $F : [0, 1] \rightarrow \mathbb{R}$ by

$$F(x) := \int_{[0, x]} f \, d\lambda$$

(a) Show that F is continuous on $[0, 1]$.

(b) Suppose that $f_n \in \mathcal{L}^1([0, 1], \lambda)$ is a sequence such that $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ and define

$$F_n(x) := \int_{[0, x]} f_n \, d\lambda$$

Show that $F_n \rightarrow F$ uniformly on $[0, 1]$.

4. Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) = 1$. Given $f \in \mathcal{L}^1(X, \mathcal{S}, \mu)$, set

- $a = \int_X f \, d\mu$.
- $s = \sqrt{\int_X |f - a|^2 \, d\mu}$.

(a) Prove that if $g : X \rightarrow \mathbb{R}$ is measurable, then for all $k > 0$ $t > 0$ we have

$$\mu(\{x \in X : |g(x)| \geq t\}) \leq \frac{1}{t} \int_X |g| \, d\mu$$

(b) Show that for all $k > 0$

$$\mu(\{x \in X : |f(x) - a| \geq ks\}) \leq \frac{1}{k^2}$$

Hint: $\{x \in X : |f(x) - a| \geq ks\} = \{x \in X : |f(x) - a|^2 \geq k^2 s^2\}$.

5. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f_n, f : X \rightarrow \mathbb{R}$ measurable.

- (a) State Fatou's Lemma.
- (b) State the Dominated Convergence Theorem.
- (c) Derive the Dominated Convergence Theorem as a consequence of Fatou's lemma.

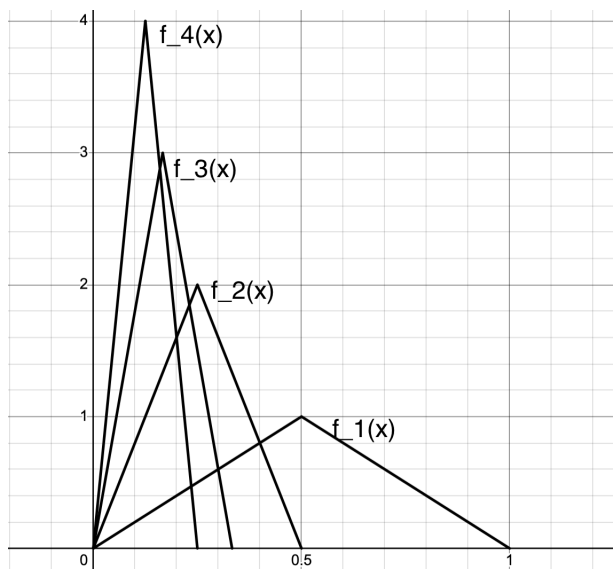
6. Compute

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x d\lambda(x)$$

You may freely use the fact that $|1 - x/n|^n \leq e^{-x}$ for all $x \in [0, n]$.

7. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be the following function:

$$f_n(x) = \begin{cases} 2n^2x & x \in [0, \frac{1}{2n}] \\ -2n^2x + 2n & x \in (\frac{1}{2n}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$



- Does there exist a function f so that $f_n \rightarrow f$ pointwise?
- Does there exist a function f so that $f_n \rightarrow f$ pointwise almost everywhere?
- Does there exist a function f so that $f_n \rightarrow f$ uniformly?
- Does there exist a function f so that $f_n \rightarrow f$ in $\|\cdot\|_1$?

8. Let (X, S, μ) be a measure space.

- State Egoroff's theorem.
- Let $f_n, f : X \rightarrow \mathbb{R}$ be measurable $f_n \rightarrow f$ pointwise and $\mu(X) < \infty$. Show that there exists a sequence of measurable sets $\{E_k\}_{k=1}^\infty$ so that
 - $\mu(X \setminus \bigcup_{k=1}^\infty E_k) = 0$, and
 - $f_n \rightarrow f$ uniformly on E_k .