Master's Comprehensive Exam
Statistical Inference

February 9, 2019

Instructions

Solve six out of the nine problems given below. Clearly indicate which 6 problems you would like to be graded on the third page. Please number your pages and start each problem at the top of a new sheet. Submit problems in order. Write your name on the back of the sheets of paper with your work. You must show all your work to receive any credit.

Problems

1. Suppose X is a random sample from the normal distribution with mean μ and variance σ^2.

   (a) Transform the variable X to achieve a Z distribution, and verify your answer.

   (b) Call the variable you created in part (a) above Y. Derive the distribution of Y^2.

   (c) Find the distribution of $\sum(X_i - \bar{X})^2$.

   (d) Find $E\left[(X_i - \bar{X})^2\right]$ in two ways, verifying that your answers are equivalent.

2. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables.

   (a) Find the MLE for p, denoted $\hat{p}$.

   (b) Find the exact distribution of $\hat{p}$. Show work.

   (c) What is the asymptotic distribution of $\hat{p}$?

   (d) Find the sufficient statistic for p using

      i) The factorization theorem

      ii) The definition of sufficiency

3. Let $X_1, \ldots, X_n$ be a random sample from the Poisson(λ) distribution.

(a) Find the UMVUE for λ.

(b) Show that your answer is statistically independent of $\bar{X}$.

4. Let $X_1, \ldots, X_n$ be a random sample from the Gamma$(\theta, 1)$ distribution, with pdf

$$f(x; \theta) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}, \quad x > 0, \theta > 0.$$

   (a) Find an unbiased estimator for $\frac{1}{\theta}$ based on $\bar{X}$. Call this estimator W.

   (b) Find the asymptotic distribution of W. That is, find the distribution of $\sqrt{n}(W - \frac{1}{\theta})$.

   (c) Generate a small sample (exact) confidence interval for θ based on your answer in (a).

   (d) Generate a large sample (approximate) confidence interval for θ based on your answer in (b).

5. Let X_n be a sequence of i.i.d. random variables. Assume that $\mathbb{E}[X_i]$ is finite. Show that $\bar{X}_n$ converges to $\mathbb{E}[X_i]$ in probability.

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Consider unbiased estimation of $\theta = \mathbb{E}[X]$. Use $Y = \min(X_1, X_2, \ldots, X_n)$.

   (a) Find the UMVUE of θ.

   (b) Derive a lower bound for the variance of any unbiased estimator of θ.

7. Let $X_1, \ldots, X_n$ be a random sample from Bernoulli(θ). Determine the level-α likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

8. Suppose X is one observation from a population with pdf

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) For testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$, find the size and sketch the power function of the test that rejects H_0 if $X > c$.

   (b) Find the most powerful level α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

   (c) Is there a UMP test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$? Prove or disprove.

9. Let $X \sim \text{Gamma}(\alpha, \beta)$. Define the random variable $Y = \frac{1}{X}$. Then Y has an inverse gamma distribution and we write $Y \sim \text{Inverse-Gamma}(\alpha, \beta)$. Show that Y has the inverse gamma pdf.

   Let X with σ known and μ unknown. Show that the inverse gamma family is conjugate for σ^2. In particular, find the posterior distribution corresponding to the prior, $p(\sigma^2) \propto (\sigma^2)^{-(\alpha+1)} e^{-\beta/\sigma^2}$.