Master's Comprehensive Exam – Statistical Inference

September 15, 2018

Name: ____

Instructions: Solve seven out of the ten problems given below. Clearly indicate which 7 problems you would like to be graded. Please number your pages, put your name on every sheet of paper you turn in, and submit problems in order.

Circle seven problems chosen: 1 2 3 4 5 6 7 8 9 10

Problems

1. **Student's Theorem:**

   Let $X_1, X_2, \ldots, X_n$ be iid random variables each having a normal distribution with mean μ and variance σ^2. Define the random variables

   $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

   Then:

   a) $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2_{n-1} distribution.
   b) $\bar{X}$ and S^2 are independent.
   c) $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution.
   d) The random variable $\frac{\bar{X}-\mu}{S/\sqrt{n}}$ has a Student's t-distribution with $n-1$ degrees of freedom.

2. Let $X_1, \ldots, X_n \sim \text{iid}\, U(0, \theta)$.

   a) Find the pdf of $X_{\max}$.
   b) Show that $X_{\max}$ is a sufficient statistic for θ using the definition of sufficiency.
   c) Use the pivot method to develop a hypothesis test for $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ based on the sufficient statistic.

3. A random variable X is said to have a Weibull$(\lambda, 2)$ distribution if it has pdf

$$f(x; \lambda) = 2\lambda^2 x \exp(-\lambda^2 x^2), \quad x > 0$$

   a) Find the sufficient statistic for λ using the Neyman factorization theorem. Verify your result using the definition of sufficiency.
   b) Find the UMVUE for λ^2 (Hint: Use transformation).
   c) Use the weak law of large numbers to find a consistent estimator for λ.
   d) Show how you can use the chi-squared distribution to create small sample confidence intervals for λ based on the sufficient statistic.

4. Let $X \sim$ Binomial(n, p).
   a) State the pmf of X, and verify that it is indeed a pmf.
   b) Find the first moment of X using the definition of expected value, and verify using the binomial mgf.
   c) Give an expression for $\Lambda(X)$, the likelihood ratio test (LRT) statistic for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$, and indicate the rejection region for the LRT.

5. Let $X_1, \dots, X_n$ be iid Poisson(λ). Assume the prior distribution of λ is Gamma(α, β) with pdf

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

   Find the Bayes estimator of λ under squared error loss.

6. The Laplace distribution may be defined as a location-scale model as follows:

$$f(x; \mu, b) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$$

   We write $X \sim$ Laplace(μ, b).
   a) Show that $\frac{X-\mu}{b}$ is a pivotal quantity.
   b) Suppose b is known. Find a 95% confidence interval for μ based on a single observation X.
   c) Now suppose $X_1, \dots, X_n$ are iid Laplace(μ, b). Find the maximum likelihood estimator for μ.
   d) Show that your estimator from (c) has a gamma distribution.

7. Let X be a random variable from an F-distribution with pdf

$$f(x) = \frac{\Gamma\left(\frac{d_1+d_2}{2}\right)}{\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)} \left(\frac{d_1}{d_2}\right)^{d_1/2} \frac{x^{d_1/2-1}}{\left(1 + \frac{d_1}{d_2}x\right)^{(d_1+d_2)/2}}$$

a) Derive the pdf of $1/X$.

b) Based on the answer to part (a), identify the distribution of $1/X$.

8. Let $f(x;\theta)$ be a pdf of the form

$$f(x;\theta) = h(x)\exp\left(\eta(\theta)T(x) - A(\theta)\right)$$

Show that

$$\frac{d}{d\theta}E[T(X)] = \mathrm{Var}(T(X))\frac{d\eta(\theta)}{d\theta}$$

provided these derivatives exist, by differentiating both sides of the equality $E[T(X)] = A'(\theta)$.

9. Let $X_1, \ldots, X_n$ be a random sample from an Exponential distribution with pdf

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0$$

a) Find a sufficient statistic $T(X)$ and find the distribution of T.

b) Find the UMVUE of θ.

10. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density

$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1$$

a) Show that the family has a monotone likelihood ratio and identify the statistic.

b) Find the distribution of the statistic in (a) and state the pdf.

c) Determine the UMP test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

d) Show how to select the rejection region (a constant c) so that the test has level α. Express c in terms of θ and α.