Mastersʹ Comprehensive Exam – Statistical Inference

Name: ____________________

February 11, 2017

Instructions: Solve seven out of the ten problems given below. Clearly indicate which 7 problems you would like to be graded. Please put your name on every sheet of paper you turn in, and submit questions in order.

Circle seven problems chosen: 1 2 3 4 5 6 7 8 9 10

1. Let $X_1, \ldots, X_n$ be a random sample from a population with Bernoulli(p) distribution.
   (a) Derive the distribution of $Y = \sum X_i$.
   (b) Find $E(Y)$ using the definition of expectation.
   (c) Find Var(Y) using the moment generating function (mgf) of Y.

2. A random variable X is said to have a Pareto distribution with parameters α and x_m if it has pdf
$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, \quad x \geq x_m.$$
   (a) Verify that $f(x)$ is a pdf.
   (b) Let $X_1, \ldots, X_n$ be a random sample from the distribution above. Find the maximum likelihood estimator (MLE) for α.
   (c) Find the Cramer-Rao lower bound for estimating α.
   (d) Discuss the efficiency of the MLE.

3. (a) The Extreme Value family of distributions is denoted by EV(γ). The value of the parameter γ determines the functional form of the cdf. Show that EV(γ) is a continuous family in γ, in the sense that the cdf converges as $\gamma \to 0$.
   (b) An expression of the Weibull distribution is
$$F(x) = 1 - e^{-(x/\lambda)^k}.$$
   Use this parameterization to find the MLE for the scale parameter λ.

4. Let $X_1, \ldots, X_n$ be a random sample from the $U(0, \theta)$ distribution.

   (a) Find an unbiased estimator for θ based on $Y_n = \max(X_1, \ldots, X_n)$. Call this estimator $\hat{\theta}$.

   (b) Find the asymptotic distribution of $\hat{\theta}$.

   (c) Generate a small sample (exact) confidence interval for θ based on your answer in (a).

   (d) Generate a large sample (approximate) confidence interval for θ based on your answer in (b).

5. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean 0 and unknown variance σ^2.

   (a) Show that this distribution is a member of the regular exponential family.

   (b) Find the UMVUE for σ^2.

   (c) Determine the form of the uniformly most powerful (UMP) test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$.

   (d) Is the test found in part (c) equivalent to the test usually used for hypotheses concerning variance?

6. Let X be a discrete random variable with the geometric distribution.

   (a) Summation and differentiation can be interchangeable if the series converges uniformly on every closed bounded subinterval. Prove this uniform convergence.

   (b) Use part (a) to find $E(X)$.

7. The random variable X has pdf $f(x)$. One observation is obtained on the random variable X, and a test of H_0 versus H_1 needs to be constructed.

   (a) Find the UMP level-α test by answering the following questions:

      - Identify the rejection region for the UMP test.
      - Describe the form of this rejection region (i.e., increasing or decreasing, etc.).
      - Clearly define the significance level α.
      - Derive the Type II error probability.

8. Consider a sample $X_1, \ldots, X_n$ from a population with pdf

$$f(x; \theta, \lambda) = \frac{\theta}{\lambda^\theta} x^{\theta - 1}, \quad 0 \leq x \leq \lambda,$$

   where both θ and λ are unknown.

   (a) Find the maximum likelihood estimators for the unknown parameters.

(b) Find the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

9. Let random variables X_i satisfy

$$X_i = \mu_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where ϵ_i are independent $N(0, \sigma^2)$ random variables, μ_i are iid, and ϵ_i and μ_i are independent. Find the approximate mean and variance for $\bar{X}$.

10. Suppose that $X_1, \ldots, X_n$ are iid Poisson(λ). Consider unbiased estimation of $e^{-2\lambda}$.

(a) Construct an unbiased estimator $\delta(X)$ based on $X_1, \ldots, X_n$.

(b) Apply the Rao-Blackwell technique to $\delta(X)$ to obtain the UMVUE.

(c) Derive the Cramer-Rao lower bound for an unbiased estimator of $e^{-2\lambda}$.