Spring 2026 – Algebra Comprehensive Exam Name: ______________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. (a) Prove that if a subgroup G of S_n contains an odd permutation, then $|G|$ is even and exactly half of the elements of G are odd permutations.

   (b) Show that for $n \geq 5$, A_n is the only subgroup of index 2 in S_n. (You may use the fact that A_n is simple for $n \geq 5$.)

   (c) Give an example of a nonabelian group with more than one subgroup of index 2.

2. (a) Prove that every group of order 105 has a nontrivial proper normal subgroup.

   (b) How many isomorphism classes of abelian groups of order 360 are there? For each one give both its invariant factor decomposition and its elementary divisor composition.

3. Label each of the following statements as true or false. Justify each answer with a proof or a counterexample.

   (a) If G is a group with $|G| = n < \infty$ and $k|n$, then G has a subgroup of order k.

   (b) If G is an abelian group with $|G| = n < \infty$ and $k|n$, then G has a subgroup of order k.

   (c) If G is a group of order p^k for p prime, then G has an abelian normal subgroup of order greater than 1.

   (d) If G is a group with $|G| = n < \infty$, $k|n$ and k is a power of a prime, then G has a subgroup of order k.

4. Given a group G, define the commutator subgroup to be the subgroup generated by commutators in G. That is,

$$[G, G] := \langle aba^{-1}b^{-1} : a, b \in G\rangle.$$

   (a) Show that $[G, G]$ is a normal subgroup of G.

   (b) Show that any group homomorphism $\varphi : G \to A$ with A abelian, takes $[G, G]$ to the the trivial subgroup in A.

   (c) For $G = D_8$ the dihedral group with 8 elements, compute the subgroup $[G, G]$ and the quotient $G/[G, G]$.

   (d) Find all group homomorphisms $\varphi : D_8 \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the group of non-zero complex numbers under multiplication.

5. (a) Let G and H be finite groups. Prove that if $|G|$ and $|H|$ are relatively prime, then the only homomorphism $\phi : G \to H$ is the trivial one (i.e. $\phi(g) = e_H$ for all $g \in G$, where e_H is the indentity of H).

   (b) How many automorphisms does S_3 have? Justify your answer.

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers, where $i^2 = -1$.
   (a) Prove that R is a unique factorization domain. You may use without proof the fact that a Euclidean domain is a unique factorization domain.
   (b) Prove that 3 is irreducible in R, but that 5 is not.
   (c) Prove that the ideal $2R$ is not maximal.

7. Let $f(x) = x^6 - 1, g(x) = x^8 - 1$ in the ring $\mathbb{R}[x]$.
   (a) Give a complete factorization of f and g into irreducibles.
   (b) Calculate $\gcd(f, g)$ and express your answer as a linear combination of f and g.
   (c) For ideals A, B of a ring, we define the <u>sum</u> $A + B = \{a + b \mid a \in A, b \in B\}$ and <u>product</u> $AB = \{\sum_{i=1}^{n} a_i b_i \mid n \in \mathbb{Z}^+, a_i \in A, b_i \in B\}$. For each of the following ideals, find a monic polynomial that generates it.
      i. $(f) + (g)$
      ii. $(f)(g)$
      iii. $(f) \cap (g)$

8. (a) Prove that every finite integral domain is a field.
   (b) Let D be an integral domain and let I be a prime ideal of D that has finite index. Prove that I is a maximal ideal of D.
   (c) Give an example of an integral domain with a nonzero prime ideal that is not maximal.

9. Let R be a commutative ring with identity $1 \neq 0$. Suppose that $e, f \in R$ such that $ef = 0, e + f = 1$.
   (a) Prove that $e^2 = e$ and $f^2 = f$.
   (b) Prove that the principal ideal eR generated by e is a ring with e as its identity.
   (c) Prove that $R \simeq eR \times fR$ as rings.

10. Construct a 3×3 matrix having an eigenvalue 1 with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, an eigenvalue of -1 with corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and an eigenvalue of 2 with corresponding eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is this matrix unique, or could there be others with this property? Justify your answer.