Spring 2025 – Algebra Comprehensive Exam Name: ________________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let H and K be finite groups. Let $G = H \times K$.
   (a) If $h \in H$ has order m and $k \in K$ has order n, what is the order of (h, k) in G? Justify your answer.
   (b) How many elements of order 20 are there in the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/10\mathbb{Z})$?

2. In each item below you are given a group G with a subgroup H. Determine if H is a normal subgroup of G. Justify your answers.
   (a) G is a finite group with a unique element b of order 2; $H = \langle b \rangle$.
   (b) $G = S_4$; $H = \langle (123) \rangle$.
   (c) $G = D_{12}$, the dihedral group of order 12; H is a Sylow 2-subgroup of G.

3. (a) Prove that there are no simple groups of order 105.
   (b) How many isomorphism classes of abelian groups of order 360 are there? For each one give both its invariant factor decomposition and its elementary divisor decomposition.

4. Let G be a group, and let H be a normal subgroup of G, and let K be any subgroup of G.
   (a) Prove that $HK = \{ hk \mid h \in H, k \in K \}$ is a subgroup of G.
   (b) Now suppose further that H has index p, where p is a prime number. Prove that either K is a subgroup of H or $[K : K \cap H] = p$.

5. Find all automorphisms of the group $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Your answer should include the following:
   (a) Describe each automorphism; that is, say what $\pi(a, b)$ is for each automorphism π and each $(a, b) \in \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.
   (b) Prove that each function you describe really is injective and surjective.
   (c) Prove that there are no other automorphisms.
   (d) State how many automorphisms there are in all.

Part II: Rings and Linear Algebra (Choose at least two.)

6. (a) Let R be a commutative ring with identity. Prove that R is a field if and only if the only ideals of R are $\{0\}$ and R.
   (b) Give an example of a ring that has exactly three ideals.
   (c) Show that $\mathbb{M}_2(\mathbb{R})$, the ring of 2×2 matrices with entries from the real numbers, has no nontrivial proper two-sided ideals but is not a division ring.

7. Let $\phi : R \to S$ be a homomorphism of rings, I an ideal of R, J an ideal of S.
   (a) Prove that $\phi^{-1}(J)$ is an ideal of R.
   (b) Prove that if ϕ is <u>surjective</u>, then $\phi(I)$ is an ideal of S.
   (c) Give an example to show that the previous part need not be true if ϕ is not surjective.

8. Let $R = \mathbb{Z}\left[\sqrt{-3}\right]$.
   (a) Prove that the elements $1 + \sqrt{-3}, 1 - \sqrt{-3}$, and 2 are all irreducible in R.
   (b) Prove that R is not a unique factorization domain.
   (c) Prove that $R = \mathbb{Z}\left[\sqrt{-3}\right]$ is not isomorphic to $S = \mathbb{Z}\left[\sqrt{-2}\right]$.

9. (a) Let R be a PID. If I is a nonzero prime ideal of R, prove that I is maximal.
   (b) Give an example of an integral domain R with a nonzero prime ideal that is not maximal. Justify your answer.
   (c) In $\mathbb{Z}_2[x]$, is the ideal generated by $x^3 + 1$ a prime ideal? Justify your answer.
   (d) Consider the quotient ring

   $$\mathbb{Z}_2[x] \big/ \left(x^3 + 1\right) = \left\{ a + bX + cX^2 \quad a, b, c \in \mathbb{Z}_2 \right. ,$$

   where X is the residue of x modulo $\left(x^3 + 1\right)$.
   i. List all the units of R.
   ii. List all zero-divisors of R.
   iii. List all ideals of R. Which of them are prime?

10. (a) Let A be an $n \times n$ matrix with real entries having eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ respectively. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
    (b) Let A be the 5×5 matrix whose entries are all 1, that is,

    $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

    i. Determine the eigenvalues for A.
    ii. Determine bases for the eigenspaces of A.
    iii. Is A diagonalizable? If so, give a diagonal matrix similar to A.