Fall 2025 – Algebra Comprehensive Exam Name: ________________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. (a) Let G be a group with normal subgroups H and K such that $H \cap K = \{1\}$. Prove that $H \times K$ is isomorphic to HK.

   (b) Give an example to show that the result of part (a) is false if only one of the subgroups is normal.

2. (a) Prove that a group of order 48 has a normal subgroup of either order 16 or order 8.

   (b) Classify all abelian groups of order $576 = 2^6 \cdot 3^2$ up to isomorphism.

3. Show that if $p < q < r$ are prime numbers and G is a group of order pqr, then at least one of the Sylow subgroups of G is normal.

4. Let G be a group acting on a set A.

   (a) Prove that for any $a \in A$, the orbit of a has order $|G : G_a|$, where G_a is the stabilizer of a in G.

   (b) Use this to prove that for any $g \in G$, the order of the conjugacy class of g is $|G : C_G(g)|$, where $C_G(g)$ is the centralizer of g in G.

   (c) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 1 & 2 & 3 & 7 & 6 \end{pmatrix} \in S_7$. Determine the sizes of the conjugacy class of σ and the centralizer $C_{S_7}(\sigma)$.

5. Let G be a group. Define the <u>commutator subgroup</u> of G to be the subgroup G' generated by all elements of the form $aba^{-1}b^{-1}$, where $a, b \in G$.

   (a) Prove that G' is a normal subgroup of G.

   (b) Prove that G/G' is abelian.

   (c) Prove or give a counterexample to the statement that if $\phi : G \to G$ is an automorphism, then $\phi(G') \subseteq G'$.

   (d) Find the commutator subgroup of D_n, the dihedral group of order $2n$.

Part II: Rings and Linear Algebra (Choose at least two.)

6. (a) Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree ≥ 1. The *Rational Root Theorem* states that if $a \in \mathbb{Q}$ with $f(a) = 0$, then $a \in \mathbb{Z}$. Prove this theorem.

   (b) In the ring $\mathbb{Z}[x]$, let A be the principal ideal generated by $x - 1$ and let B be the principal ideal generated by $x + 1$.

      i. Show that $A \cap B$ is a principal ideal.

      ii. Show that $A + B$ is a maximal ideal that is not principal.

7. (a) Show that $R = \mathbb{Z}\left[\sqrt{-2}\right]$ is a Euclidean domain under the usual norm, $N\left(x + y\sqrt{-2}\right) = x^2 + 2y^2$. In other words, show that given $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ and $N(r) < N(b)$. (You may use without proof the fact that $N(ab) = N(a)N(b)$.)

   (b) The result in (a) is not true if -2 is replaced by -3. Circle the line of your work for (a) that would fail if we replaced -2 by -3.

   (c) Show carefully that $3 + \sqrt{2}$ is irreducible in $R = \mathbb{Z}\left[\sqrt{2}\right]$.

8. (a) Prove that every Euclidean domain is a PID.

   (b) Prove that in a PID, every ascending chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

   stabilizes; that is, there is a value n for which $I_n = I_{n+1} = \cdots$. (Hint: Show that the union of all ideals in the chain is an ideal.)

   (c) Give a proof or a counterexample to the following statement. Let R be a PID. If $r \in R$ is irreducible, then (r) is a maximal ideal.

9. Let R be an integral domain. Label each of the following statements as true or false. Justify each answer with a proof or counterexample.

   (a) For any two nonzero proper ideals I, J of R, $I \cap J \neq \{0\}$.

   (b) Every nonzero prime ideal of R is maximal.

   (c) Every prime element of R is irreducible.

   (d) If R is finite, then R is a field.

10. Let V be a finite-dimensional vector space. Let $T : V \to V$ be a linear transformation with $T \circ T = T$.

   (a) What are the possible eigenvalues for T?

   (b) Let $v \in V$ and set $w_0 = (I - T)(v)$ and $w_1 = T(v)$ (where I is the identity map). Show that $T(w_0) = 0, T(w_1) = w_1$, and $v = w_0 + w_1$.

   (c) Show that $W = \{v : T(v) = v\}$ is a subspace of V and that $\ker T \cap W = \{0\}$.

   (d) Show that there exists a basis $\{u_1, u_2, \ldots, u_n\}$ of V and an integer $k \leq n$ such that

$$T\left(a_1u_1 + a_2u_2 + \cdots + a_nu_n\right) = a_1u_1 + \cdots + a_ku_k.$$