Sample Applied Nonlinear ODE Comprehensive Exam

Do any 6 problems. Clearly indicate which ones you want to be graded. Good luck!

Problems	1	2	3	4	5	6	7	8
Select								

1. Consider $\dot{x} = y$ and $\dot{y} = f(x, \lambda)$, where f and f' are continuous.

   (a) Show that the index I_Γ of any simple closed curve Γ that encloses all equilibrium points can only be 1, -1 or zero.

   (b) Show that at a bifurcation point the sum of the indices of the equilibrium points resulting from the bifurcation is unchanged.

   (c) By using the results in (a) and (b), deduce that the system $\dot{x} = y$, $\dot{y} = -\lambda x + x^3$ has a saddle point at $(0, 0)$ when $\lambda < 0$, which bifurcates into a center and two saddle points as λ becomes positive.

2. Find the equivalent linear equation and the frequency-amplitude relation for the equation
$$\ddot{x} + \operatorname{sgn}(x) = 0, \text{ where } \operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}.$$

3. Apply the Lindstedt method to van der Pol's equation $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$, $|\varepsilon| \ll 1$. Determine the frequency of the limit cycle to the order of ε^2.

4. Let $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ be a regular autonomous system of dimension n, with $\mathbf{X}(\mathbf{0}) = \mathbf{0}$. Suppose there exists a function $U(\mathbf{x})$ such that in some neighborhood $\|\mathbf{x}\| \le k$, where $k > 0$,

   (i) $U(\mathbf{x})$ and its partial derivatives are continuous;

   (ii) $U(\mathbf{0}) = 0$;

   (iii) $\dot{U}(\mathbf{x})$ is positive definite for the given system;

   (iv) in every neighborhood of the origin there exists at least one point $\mathbf{x}$ at which $U(\mathbf{x}) > 0$.

   Show that the zero solution of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is unstable.

5. (a) Suppose that, for the two plane systems $\dot{\boldsymbol{x}}_1 = \mathbf{X}_1(\boldsymbol{x}_1), \dot{\boldsymbol{x}}_2 = \mathbf{X}_2(\boldsymbol{x}_2)$, and for a given closed curve Γ, there is no point on Γ at which $\mathbf{X}_1$ and $\mathbf{X}_2$ are opposite in direction. Show that the index of Γ is the same for both systems.

   (b) The system $\dot{x} = y,\ \dot{y} = x$ has a saddle point at the origin. Show that the index of the origin for the system $\dot{x} = y + cx^2y,\ \dot{y} = x - cxy^2$ is likewise -1 for $c \neq 0$.

6. (a) Let the origin be the equilibrium point of the system $\dot{x} = ax + by + h_1(x, y), \dot{y} = cx + dy + h_2(x, y)$ where a, b, c, d are constants, and h_1, h_2 are differentiable and their first derivatives are continuous. Moreover

$$h_1(x, y) = O(x^2 + y^2), \quad h_2(x, y) = O(x^2 + y^2), \quad \text{as } x^2 + y^2 \to 0.$$

Then show that the zero solution of the system is asymptotically stable when its linear approximation is asymptotically stable.

(b) Test the stability of the zero solution of the van der Paol's equation

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0,$$

when $\beta < 0$.

7. (a) The n-dimensional system $\dot{\boldsymbol{x}} = -\text{grad } W(\boldsymbol{x})$ has an isolated equilibrium point at $\boldsymbol{x} = 0$. Show that the zero solution is asymptotically stable if W has a local minimum at $\boldsymbol{x} = 0$. Given a condition for instability of zero solution and justify.

(b) Determine the stability of the zero solution of the system

$$\dot{x} = x^2 - y^2, \qquad \dot{y} = -2xy.$$

8. Given the equations $\dot{x} = \mu x + y - xf(r), \dot{y} = -x + \mu y - yf(r)$, where $r = \sqrt{x^2 + y^2}$, $f(r), f'(r)$ are continuous for $r \geq 0$, $f(0) = 0$, $f'(r) > 0$ for $r > 0$, and $f(r) \to \infty$ as $r \to \infty$. The origin is the only equilibrium point. Then prove the followings:

(i) for $\mu < 0$ the origin is a stable spiral covering the whole plane;

(ii) for $\mu = 0$ the origin is a stable spiral;

(iii) for $\mu > 0$ there is a stable limit cycle whose radius increases from zero as μ increases from zero.