Sample Applied Nonlinear ODE Comprehensive Exam

Do any six problems. Clearly indicate in the table below which problems you want to be graded. If you do not select any problems we will grade the first 6 problems. Good luck!

Problems	1	2	3	4	5	6	7	8
Select								

1. (a) Show that the index of the point at infinity, I_∞, for the system $\dot{x} = X(x, y), \dot{y} = Y(x, y)$, having a finite number n of equilibrium points with indices $I_i,\ i = 1, 2, \ldots, n$, is given by $I_\infty = 2 - \sum_{i=1}^{n} I_i$.

   (b) Find the index I_∞ of the system $\dot{x} = 2xy,\ \dot{y} = x^2 - y^2$.

2. Find the equivalent linear equation for $\ddot{x} + \varepsilon(|x| - 1)\dot{x} + x - \varepsilon x = 0$, and obtain the resulting frequency and amplitude approximations of the limit cycle.

3. Use the Lindstedt's method to find the $O(1)$ and $O(\varepsilon)$ terms in the expansion of the periodic solutions of $\ddot{x} + x + \varepsilon x^2 = 0$. Also find the frequency-amplitude relation for the periodic solutions up to $O(\varepsilon^2)$.

4. (a) Show that all solutions of the regular linear system $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t)$ have the same Liapunov stability property, and the Liapunov stability of those solutions is the same as that of the zero solution of the homogeneous equation $\dot{\boldsymbol{\xi}} = \mathbf{A}(t)\boldsymbol{\xi}$.

   (b) For the regular linear system $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x}$, show that all solutions are Liapunov stable on $t \geq t_0,\ t_0$ arbitrary, if and only if every solution is bounded as $t \to \infty$.

5. (a) If $\boldsymbol{h}(\mathbf{0}, t) = \mathbf{0}$, A is constant, and

   (i) the solutions of $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ are asymptotically stable;

   (ii) $\displaystyle\lim_{||\boldsymbol{x}|| \to 0} \frac{||\boldsymbol{h}(\boldsymbol{x}, t)||}{||\boldsymbol{x}||} = 0$ uniformly in t, $0 \leq t < \infty$;

   show that the zero solution $\boldsymbol{x}(t) = \mathbf{0}$ for $t \geq 0$, is an asymptotically stable solution of the regular system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}, t)$.

   (b) Test the stability of the zero solution of the system

$$\dot{x} = -3x + y + \frac{x^2}{1+t}, \qquad \dot{y} = 2x - y + \frac{xy}{1+t}.$$

6. Let the origin be an equilibrium point of the regular two-dimensional system $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})$ where $\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} h_1(\boldsymbol{x}) \\ h_2(\boldsymbol{x}) \end{bmatrix}$, and $h_1(\boldsymbol{x}), h_2(\boldsymbol{x}) = O(|\boldsymbol{x}|^2)$ as $|\boldsymbol{x}| \to 0$. Suppose the eigenvalues of $\mathbf{A}$ are both nonzero, real, and one of them is positive. Show that the zero solution of the system is unstable.

7. (a) Show that the Liénard equation $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$, or the equivalent system $\dot{x} = y,\ \dot{y} = -f(x, y)y - g(x)$, has at least one periodic solution if f and g are continuous, and they satisfy the following conditions:
   (i) there exists $a > 0$ such that $f(x, y) > 0$ when $x^2 + y^2 > a^2$;
   (ii) $f(0, 0) < 0$;
   (iii) $g(0) = 0,\ g(x) > 0$ when $x > 0$, and $g(x) < 0$ when $x < 0$;
   (iv) $G(x) = \int_0^x g(u)\ du \to \infty$ as $x \to \infty$.

   (b) Apply the result in (a) to show that $\ddot{x} + (|x| + |\dot{x}| - 1)\dot{x} + x|x| = 0$ has at least one periodic solution.

8. (a) Let $\mathcal{C}$ be a closed path for the system $\dot{\boldsymbol{x}} = \mathbf{X}(\boldsymbol{x})$, having $\mathcal{D}$ as its interior. Show that

$$\iint_{\mathcal{D}} \operatorname{div}(\mathbf{X})\ dxdy = 0.$$

   (b) Assume that van der Pol's equation in the phase plane

$$\dot{x} = y, \qquad \dot{y} = -\varepsilon(x^2 - 1)y - x$$

   has a single closed path, which, for ε is small, is approximately a circle, center the origin, of radius a. Use the result in part (a) to show that approximately

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2 - 1)\ dydx = 0,$$

   and so deduce a.