Sample Applied Nonlinear ODE Comprehensive Exam

Do any six problems. Clearly indicate in the table below which problems you want to be graded. If you do not select the problems we will grade the first 6 problems. Good luck!

Problems	1	2	3	4	5	6	7	8
Select								

1. (a) Show that the phase paths of the Hamilton system $\dot{x} = -\partial H/\partial y, \dot{y} = \partial H/\partial x$ are given by $H(x,y) = constant$. If (x_0, y_0) is an equilibrium point, show that (x_0, y_0) is stable according to the linear approximation if $H(x,y)$ has a maximum or a minimum at the point. Assume that all the second derivatives of H are nonzero at x_0, y_0.

   (b) For the system $\dot{x} = y[16(2x^2+2y^2-x)-1]$ and $\dot{y} = x-(2x^2+2y^2-x)(16x-4)$, show that the system is Hamiltonian and obtain the Hamiltonian function $H(x,y)$. Obtain the equilibrium points and classify them.

2. (a) For the system $\dot{x} = X(x,y),\ \ \dot{y} = Y(x,y)$, show that there are no closed paths in a simply connected region in which $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y}$ is of one sign.

   (b) Use part (a) and the directions of the phase paths to show that the system $\dot{x} = xy - y^2,\ \ \dot{y} = x^3 y + \sin x$ has no closed path in the entire x, y plane.

3. Use the method of equivalent linearization to find the amplitude and frequency of the limit cycle of the equation $\ddot{x} + \varepsilon(|x|-1)\dot{x} + x + \varepsilon x^3 = 0,\ 0 < \varepsilon \ll 1$. State the equivalent linear equation.

4. Apply Lindstedt's method to the problem

$$(1+\varepsilon\dot{x})\ddot{x} + 4x = 0,\ x(\varepsilon,0) = a,\ \dot{x}(\varepsilon,0) = 0,$$

   to obtain 2π-periodic solutions to $O(\varepsilon)$, and to obtain the amplitude-frequency relation to $O(\varepsilon^2)$.

5. (a) Write down the definitions of Poincaré stability and Liapunov stability for plane autonomous systems.

   (b) Prove that Liapunov stability of a solution implies Poincaré stability for plane autonomous systems. Construct an autonomous system as a counter-example and show that the converse is not true.

6. (a) Suppose that

   (i) $\mathbf{A}$ is a constant $n \times n$ matrix whose eigenvalues have negative real parts;

   (ii) For $t_0 \leq t < \infty$, $\mathbf{C}(t)$ is continuous and $\displaystyle\int_{t_0}^{t} ||\mathbf{C}(t)||\, dt$ is bounded.

   Show that all solutions of the linear homogeneous system $\dot{\mathbf{x}} = \{\mathbf{A} + \mathbf{C}(t)\}\mathbf{x}$ are asymptotically stable.

   (b) Use the result in (a) to investigate the stability of the solutions of the linear system

$$\dot{x} = (-2 + te^{-t})x + y + t^2 \sin(t), \;\; \dot{y} = -5x + \sqrt{3}te^{-t}y - \pi t.$$

7. (a) Suppose that in a neighborhood $\mathcal{N}$ of the origin, the regular system $\dot{\boldsymbol{x}} = \mathbf{X}(\boldsymbol{x})$ and the function $V(\boldsymbol{x})$ satisfy

   (i) $\mathbf{X}(\mathbf{0}) = \mathbf{0}$;

   (ii) $V(\boldsymbol{x})$ is continuous and positive definite;

   (iii) $\dot{V}(\boldsymbol{x})$ is continuous and negative semidefinite.

   Show that the zero solution of the system is uniformly stable.

   (b) Show the zero solution of the system $\dot{x} = y - \sin^3 x, \quad \dot{y} = -4x - \sin^3 y$ is uniformly stable by using the function $V(x, y) = x^2 + \alpha y^2$ with a suitable α.

8. (a) State the Poincaré-Bendixson theorem. Do NOT prove the theorem.

   (b) Consider the system

$$\dot{r} = r(3 - r^2) + \mu r \cos\theta, \quad \dot{\theta} = -1,$$

   where $r^2 = x^2 + y^2$ and $\mu > 0$ is a parameter. Check the conditions of the Poincaré-Bendixson Theorem and determine for what values of μ there exists a periodic solution. Find the region where the closed path is in.