MASTER'S COMPREHENSIVE EXAMINATION
STATISTICAL INFERENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
SPRING SEMESTER, 2015

Directions: Work 7 out of the following 10 problems.

1. Prove Student's Theorem: Let $X_1, \ldots, X_n$ be iid random variables each having a normal distribution with mean μ and variance σ^2. Define the random variables

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

then

a) $\overline{X}$ has a $N(\mu,\sigma^2/n)$ distribution.
b) $\overline{X}$ and S^2 are independent
c) $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.
d) The random variable $T = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ has a Student's t-distribution with $n-1$ degrees of freedom.

2. Let $X_1, \ldots X_n$ be a random sample from a distribution with pdf
$f_X(x;\theta) = \theta^2 x e^{-\theta x}$, $x > 0$, $\theta > 0$.
a) Find the MLE for θ.
b) Find the Rao-Cramer lower bound for estimating θ.
c) Is the MLE from part a. efficient? Show your work.

3. A random variable X is said to have a $Weibull(\lambda,2)$ distribution if it has pdf
$f_X(x) = \dfrac{2x}{\lambda^2} e^{-(x/\lambda)^2}$, $x \geq 0$
.
a) Find the sufficient statistic for λ using the Neyman factorization theorem. Verify your result using the definition of sufficiency.
b) Find the UMVUE for λ^2 (HINT: Use transformation).
c) Use the weak law of large numbers to find a consistent estimator for λ.
d) Use the delta method to derive the asymptotic distribution of a scaled version of your answer to c.

4. Let $X_1 \sim Gamma(1,2)$ and $X_2 \sim Gamma(2,2)$.
a) Find the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$.
b) Find the marginal pdf of Y_2. What kind of distribution is this?
c) Find $E(Y_2)$.

5. Verify that the distribution of the likelihood ratio statistic under normal $N(\mu,\sigma^2)$ sampling is exactly $\chi^2(1)$.

6. We are interested in randomizing one or more of the parameters in a parametric probability model. Consider randomizing the success parameter in the Bernoulli trials. Suppose that P_i has the beta distribution on the interval (0, 1) with positive parameters (α, β), which is known to be left and right parameters. Suppose that $X = (X_1, X_2, \ldots X_n)$ is a sequence of indicator random variables with the property that X is a conditionally independent sequence given $P_i = p_i$, with

$$P(X_i = 1 \mid P_i = p_i) = p_i, p_i \in (0,1), i = 1,2, \ldots, n.$$

Now let the number of successes in the first n trials be

$$Y_n = \sum_{i=1}^{n} X_i$$

Then,

a) Find the expectation of Y_n and variance of Y_n.
b) Find the joint pdf of P_i and Y_n.
c) Find the marginal distribution of Y_n.

7. Let X be a random variable and $M_X(t)$ be the moment generating function (mgf) of X.

a) Show that $M_X(t) \geq \exp[tE(X)]$, for all t for which the mgf is defined.
b) Prove that $P(X \geq 0) \leq M_X(t)$, for all $t \geq 0$ for which the mgf is defined.
c) What are general conditions on a function $h(t, x)$ such that

$$P(X \geq 0) \leq E[h(t, X)],$$

for all $t \geq 0$, for which $E[h(t, X)]$ exists.

8. Color blindness (or, more accurately, color vision deficiency) is an inherited condition that affects males more frequently than females. According to Prevent Blindness America, an estimated 8 percent of males and less than 1 percent of females have color vision problems.

a) How large must a sample be if the probability of it containing a color-blind male is to be .95 or more?
b) How large must a sample be if the probability of it containing a color-blind female is to be .95 or more?

9. Consider a sequence of RVs defined on the same probability space where

$$P(X_n = 1) = \frac{1}{n} = 1 - P(X_n = 0)$$

a) Show that X_n converges in distribution to a random variable X. Also find the CDF of X.
b) Show that X_n converges in probability to a random variable X.

10. Let $X_1, \ldots, X_m;\ Y_1, \ldots, Y_n$ be iid as $U(0, \theta)$ and $U(0, \varphi)$, respectively. If $n > 1$, determine the UMVUE of θ/φ.