Instructions: Solve *seven* out of the *ten* problems given below. Clearly indicate which 7 problems you would like to be graded. Please put your name on every sheet of paper you turn in, and submit questions in order.

Circle *seven* problems chosen: 1 2 3 4 5 6 7 8 9 10

September 12, 2015

1. Let $X_1, X_2, \ldots$ be iid with the uniform distribution on [0, 1]. Find the number φ so that
$$\sqrt{n}\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n} - \varphi\right]$$
converges in distribution, and identify the limiting distribution.
(Hint: Define $Y_i = \log X_i$, $i = 1, \ldots, n$ and apply the CLT on $Y_1, \ldots, Y_n$ and the δ-method.)

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with density
$$f(x|\theta) = e^{-(x-\theta)}, \qquad x > \theta.$$
   a. Show that the family has a monotone likelihood ratio and identify the statistic $T(\boldsymbol{X})$.
   b. Find the distribution of $T(\boldsymbol{X})$.
   c. Determine the UMP test for testing $H_0: \theta \le \theta_0$ versus $H_1: \theta > \theta_0$. Make sure you show how to select the rejection region (a constant c) so that the test has level α. In other words, express c in terms of θ_0 and α.

3. Let $X_1, X_2, \ldots, X_n$, be a random sample from the distribution with pdf,
$$f(x; \theta) = \theta(1+x)^{-(1+\theta)}, \qquad x > 0.$$
   a. Assuming that the parameter space is $\Theta = (1, \infty)$, find a method of moments estimator of θ.
   b. Derive the maximum likelihood estimator of θ^{-1}.
   c. Find the Cramer-Rao lower bound for the variance of unbiased estimators of θ^2.
   d. Find the UMVUE of θ. Be sure to justify your answer.

4. A random variable X is said to have a *Pareto* distribution with parameters and if it has pdf
$$f(x) = \alpha\beta(1+\beta x)^{-\alpha-1}, \qquad 0 \le x < \infty, \alpha > 0, \beta > 0.$$
   a. Verify that $f(x)$ is a pdf.
   b. Let $X_1, \ldots, X_n$ be a random sample from the distribution above. Find the maximum likelihood estimator, $\hat{\alpha}$ for α.
   c. Find the Cramer-Rao lower bound for estimating α.
   d. Discuss the efficiency of $\hat{\alpha}$.

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5. A homogeneous stick of length L is broken at two points. What is the probability that a triangle can be formed from the three pieces?

6. Prove or disprove that the random variables $R = \sqrt{X_1^2 + X_2^2}$ and $W = \tan^{-1}(X_1/X_2)$ are independent.

7. Let $X_1, \dots, X_{n_1}$ be a random sample from the $N(\mu_1, \sigma^2)$ and let $Y_1, \dots, Y_{n_2}$ be a random sample from the $N(\mu_2, \sigma^2)$ distribution which is independent of the first random sample. Consider the likelihood ratio test for testing $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$.
   a. Find the unrestricted MLE's of μ_1, μ_2 and σ^2.
   b. Find the MLE's of $\mu = \mu_1 = \mu_2$ and σ^2 under H_0.
   c. Show that
   $$\sum_{i=1}^{n_1}(x_i - \hat{\mu})^2 + \sum_{i=1}^{n_2}(y_i - \hat{\mu})^2 = \sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2 + n_1(\bar{x} - \hat{\mu})^2 + n_2(\bar{y} - \hat{\mu})^2.$$
   d. Show that the likelihood ratio test is
   $$\phi(x) = \begin{cases} 1, & \text{if } F > c \\ 0, & \text{if } F < c, \end{cases}$$
   $P_{H_0}(F > c) = \alpha$, where
   $$F = \frac{n_1(\bar{x} - \hat{\mu})^2 + n_2(\bar{y} - \hat{\mu})^2}{\left[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2\right]/(n_1 + n_2 - 2)}.$$

8. Determine the better method for estimating p^2, the square of the Bernoulli probability of success by completing the following.
   a. If $X_1, \dots, X_n \sim iid\ Ber(p^2)$, find the MLE for p^2. Is it unbiased?
   b. If $Y_1, \dots, Y_n \sim iid\ Ber(p)$, find the MLE for p^2. Is it unbiased?
   c. Both MLE's involve sample means. Find the asymptotic distribution of an appropriately normalized version of each.
   d. Compare the asymptotic variances. For which values of p is the estimator found in part a) more efficient?

9. Let $X_1, \dots, X_n$ be a random sample from the $Unif(-\theta, 0)$ distribution, with $\theta > 0$.
   a. Find an unbiased estimator for θ based on $X_{(1)} = \min(X_1, \dots, X_n)$.
   b. Generate a small sample (exact) confidence interval for θ based on your answer in a).
   c. Find the asymptotic distribution of $Y_n = n\left(X_{(1)} + \theta\right)$. That is, find the distribution of Y, where $Y_n \xrightarrow{D} Y$.

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10. Let $X_1, \ldots, X_{25}$ be a random sample from $N(\mu, 100)$.
    a. Show that this is a member of the regular exponential class
    b. Determine the uniformly most powerful test of $H_0: \mu = 75$ vs. $H_1: \mu > 75$ at the $\alpha = 0.10$ level of significance. Be sure to justify your answer.
    c. Find the power function for the test you found in b). Determine where it reaches a minimum.

STOP HERE!!