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A NOTE ON THE SUPPORT OF A SOBOLEV FUNCTION ON A k-CELL

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ABSTRACT. It is shown that a k-cell (the homeomorphic image of a closed ball in $\mathbb{R}^k$) in $\mathbb{R}^n$, $1 \le k < n$, cannot support a function in $W^{1,p}(\mathbb{R}^n)$ if $p > [\frac{k+1}{2}]$, the greatest integer in $(k+1)/2$.

1. INTRODUCTION

In this paper we investigate the question of determining whether the homeomorphic image of a k-dimensional closed ball in $\mathbb{R}^n$, $1 \le k < n$, a k-cell, can support a Sobolev function $f \in W^{1,p}(\mathbb{R}^n)$. Since a k-cell is nowhere dense in $\mathbb{R}^n$, it is natural to first inquire whether a compact, nowhere dense set can support a Sobolev function. Of course, this question is only of interest when the set has positive Lebesgue measure. For the case $p > n$, the answer is obvious, since any function of $W^{1,p}(\mathbb{R}^n)$ has a continuous representative in $\mathbb{R}^n$, and a nonzero continuous function cannot have its support on a nowhere dense compact set. However, for the case $1 < p \le n$, Polking [Pol72, Theorem 4] showed that there is a nonzero element of $W^{1,p}(\mathbb{R}^n)$ that does have nowhere dense compact support. A characterization of nowhere dense sets that can support $W^{1,p}(\mathbb{R}^n)$ functions in terms of capacity is given in [AH96, Theorem 11.3.2]. The existence of homeomorphisms that carry sets of Lebesgue measure zero into sets of positive measure is well known. Besicovitch [Bes50] constructed a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^3$ that carries null sets onto sets of positive measure. In [Pon87], a homeomorphism in $W^{1,q}(\mathbb{R}^n;\mathbb{R}^n)$, with $q < n$, was constructed carrying null sets into sets of positive Lebesgue measure. The question we investigate in this paper is whether a k-cell in $\mathbb{R}^n$, $0 < k < n$, can support a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$. The complete answer to this question remains an open problem. Bagby and Gauthier [BG98] proved that for $n > k > 0$ and $p > \max(1, k-1)$, only the zero function in $W^{1,p}(\mathbb{R}^n)$ has its support contained in a k-cell. Our contribution to this question is to offer an improvement of this result for $n \ge 3$. In Theorem 5 of this paper it is shown that the Bagby-Gauthier result remains true by requiring $p > [\frac{k+1}{2}]$ where $[\frac{k+1}{2}]$ denotes the greatest integer in $\frac{k+1}{2}$. The main ingredient of the proof is that under these restrictions on p, if $u \in W^{1,p}(\mathbb{R}^{k+1})$ is not identically zero, then u has a representative that is defined, continuous and strictly positive (or negative) on a pair of linked spheres of dimension $[\frac{k+1}{2}]$ and $[\frac{k}{2}]$; see Definition 1.

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2. Preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$, its s-dimensional Hausdorff measure by $H^s(E)$, and its p-capacity by $\gamma_p(E)$. We refer the reader to [MZ97, Section 2.1] for the definitions of p-capacity, its comparison to Hausdorff measure, and its relationship to functions in the Sobolev class $W^{1,p}$. In particular, we recall that

$$\gamma_p(E) = 0 \quad \text{if and only if } H^{n-p+\varepsilon}(E) = 0 \text{ for all } \varepsilon > 0 \text{ and } 1 \leq p \leq n. \tag{2.1}$$

The restriction of a function u to a set E is denoted by $u \llcorner E$. With $\Omega \subset \mathbb{R}^n$ an open set and $n \geq 1$, the Sobolev space $W^{1,p}(\Omega)$, $p \geq 1$, consists of those functions $u \in L^p(\Omega)$ for which the first-order distributional partial derivatives of u also belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is given by

$$\|u\|_{1,p;\Omega} = \left(\sum_{k=0}^{n} \int_\Omega |D^k u|^p \, dx \right)^{1/p}.$$

An alternate definition of the Sobolev space is furnished by the fact that $C^\infty(\Omega) \cap \{u : \|u\|_{1,p;\Omega} < \infty\}$ is dense in $W^{1,p}(\Omega)$. A sequence of functions that converges except on a set of γ_p zero is said to converge p-q.e. A function u is called p-quasicontinuous if for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ with $\gamma_p(U) < \varepsilon$ such that $u \llcorner \mathbb{R}^n \setminus U$ is continuous. Any function $u \in W^{1,p}(\mathbb{R}^n)$ has a representative that is p-quasicontinuous. Indeed, the pointwise limit of a suitable subsequence of smooth functions $\{u_k\}$ that converge strongly to u in $W^{1,p}$ defines a p-quasicontinuous representative; cf. [MZ97, Lemma 2.19]. Throughout, we will employ the notation $\mathbf{u}$ (boldface u) to denote a p-quasicontinuous representative of $u \in W^{1,p}(\mathbb{R}^n)$ and $B_x^n(r)$ to denote the open ball in $\mathbb{R}^n$ with center x and radius r. Recall that an arbitrary $u \in L^p(\mathbb{R}^n)$ has an L^p-Lebesgue point almost everywhere; that is,

$$\lim_{r \to 0} \frac{1}{|B_a^n(r)|} \int_{B_a^n(r)} |u(x) - u(a)|^p \, dx = 0$$

for almost all $a \in \mathbb{R}^n$. When $u \in W^{1,p}(\mathbb{R}^n)$, this limit is zero for all a in the complement of a γ_p null set. If a is a Lebesgue point for u and if $\{u_k\}$ is taken as the standard mollifiers of u, then $u_k(a) \to u(a)$. We will use the notation $\bar{u}(a, r)$ to denote the integral average of u over the the ball $B_a^n(r)$, and $\bar{u}(a) := \lim_{r \to 0} \bar{u}(a, r)$ when the limit exists. Likewise, we let $\overline{\nabla u}(a)$ denote the value of ∇u in terms of the limit of its integral averages at a.

Throughout, we will assume that $1 \leq p \leq n$ since our problem becomes trivial if $p > n$. We will make extensive use of the "coarea formula", stated below.

Theorem 1 ([Fed59, Theorem 3.1])**.** *If X and Y are separable Riemannian manifolds of class 1 of respective dimensions m and k, $m \geq k$, and $f \colon X \to Y$ is a Lipschitzian map, then*

$$\int_X g(x) Jf(x) \, dH^m(x) = \int_Y \int_{f^{-1}(y)} g(x) dH^{m-k}(x) dH^k(y) \tag{2.2}$$

whenever $g \colon X \to \mathbb{R}^1$ is H^m integrable. Here, $Jf(x)$ denotes the square root of the sum of the squares of the determinants of the $k \times k$ minors of the differential of f at x.

We will not need the full strength of Federer's coarea formula, but merely the case when X and Y are subsets of Euclidean space.

3. Linked spheres in $\mathbb{R}^n$

Definition 1. With S^k denoting the standard k-sphere in $\mathbb{R}^{k+1}$, let $\Sigma_1^k := h_1(S^k)$ and $\Sigma_2^{n-1-k} := h_2(S^{n-1-k})$ be the images of disjoint topological embeddings, h_1, h_2, of S^k and S^{n-1-k} into $\mathbb{R}^n$. The linking number of Σ_1^k and Σ_2^k is defined as the topological degree of the mapping

$$S^k \times S^{n-1-k} \xrightarrow{f} S^{n-1} \tag{3.1}$$

defined by $f(x,y) = \dfrac{h_1(x) - h_2(y)}{|h_1(x) - h_2(y)|}$; see [Hir76] or [Rol76].

Remark 1. Recall that the topological degree is defined to be that integer, $\deg(f)$, so that the induced homomorphism of homology groups, $f_*\colon H_{n-1}(S^k \times S^{n-1-k}) \to H_{n-1}(S^{n-1})$, is given by multiplication by $\deg(f)$. Note that both homology groups are isomorphic to $\mathbb{Z}$. Recall also that if f is smooth, then $\deg(f) = \sum_{x \in f^{-1}(y)} \mathcal{J} f(x)$ where y is a regular value of f, and where $\mathcal{J} f(x)$ denotes the Jacobian of f at x.

Theorem 2. *Let $\bar{B}^{n-1}$ be a closed ball in $\mathbb{R}^{n-1}$ and suppose that $h\colon \bar{B}^{n-1} \to \mathbb{R}^n$ is an embedding, with disjoint $\Sigma_1^k, \Sigma_2^{n-1-k} \subset h(B^{n-1})$. Then the linking number of Σ_1^k and Σ_2^{n-1-k} is* 0.

Proof. The mapping f in (3.1) can be factored as $f = f_2 \circ f_1$ where

$$f_1\colon S^k \times S^{n-1-k} \to \left(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) \times \Sigma_2^{n-1-k}$$

and

$$f_2\colon \left(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) \times \Sigma_2^{n-1-k} \to S^{n-1}.$$

Let $H_i(K)$ denote the *ith* homology group of K. Recalling the Künneth theorem, [Mas91, Section XI.4, Theorem 4.1], and the fact that $H_i\left(S^k\right)$ and $H_i\left(S^k \backslash S^j\right)$ are torsion free, we have that

$$\begin{aligned} &H_q\left(\left(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) \times \Sigma_2^{n-1-k}\right) \\ &\quad = \sum_{j=0}^{q} H_j\left(h(\overline{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) \otimes H_{q-j}\left(\Sigma_2^{n-1-k}\right). \end{aligned}$$

Since $h(\bar{B}^{n-1} \setminus \Sigma_2^{n-1-k})$ is homeomorphic to $\bar{B}^{n-1} \setminus h^{-1}(\Sigma_2^{n-1-k})$, the complement of an embedded $(n-1-k)$-sphere, we obtain the following homology groups: for $k > 1$,

$$H_q\left(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) = \begin{cases} \mathbb{Z} & \text{when } q = 0,\ k-1, \text{ and } n-2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q\left(\Sigma_2^{n-1-k}\right) = \begin{cases} \mathbb{Z} & \text{when } q = 0, \text{ and } n-1-k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$H_q\left(\left(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}\right) \times \Sigma_2^{n-1-k}\right) = 0$$

except when

$$q \in \{0, k-1, n-2, n-2-k, 2n-3-k\}.$$

Consequently, $H_{n-1}\left(\left(h(\bar{B}^{n-1})\setminus\Sigma_2^{n-1-k}\right)\times\Sigma_2^{n-1-k}\right)=0$ unless $2n-3-k=n-1$; that is, if $k=n-2$. However, without loss of generality, it can be assumed that $k<n/2$, and therefore $H_{n-1}\left(\left(h(\bar{B}^{n-1})\setminus\Sigma_2^{n-1-k}\right)\times\Sigma_2^{n-1-k}\right)=0$ except when $n=3$. When $n=3$, the Jordan Curve Theorem can be applied to the curves Σ_1^k and Σ_2^{n-1-k} in $h(\bar{B}^2)$ to conclude that one of the curves is null homotopic in the complement of the other. Since the degree is a homotopy invariant, in this case the degree will be 0 as well. □

4. Quasicontinuous representatives on spheres

For $3\le m+2\le n$ and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, we will write $x=(x',x'')$ where $x'=(x_1,x_2,\ldots,x_{m+1})$ and $x''=(x_{m+2},\ldots,x_n)$. Let $Q\colon\mathbb{R}^n\to\mathbb{R}^{n-m-1}$ be defined as $Q(x)=x''$. Then $Q^{-1}(x'')$ is an $(m+1)$-dimensional "horizontal" affine space. Throughout, we will use the notation $S_x^m(r)$ to denote the m-sphere centered at $x\in\mathbb{R}^n$ of radius r that is contained in $Q^{-1}(x'')$. Thus,

$$\begin{aligned} S_x^m(r) &= \{y\in Q^{-1}(x''):|y-x'|=r\} \\ &= \{y\in\mathbb{R}^n: y=(y',x''),\ |y'-x'|=r\}. \end{aligned} \tag{4.1}$$

We will also consider spheres in planes orthogonal to $Q^{-1}(x'')$, using the familiar notation for spheres. Thus, for $b\in S_a^m(r)$ we will consider an $(n-m-1)$-sphere centered at b in the $(n-m)$-plane orthogonal to $Q^{-1}(a'')$ that contains the line through a and b; thus, for $b\in S_a^m(r)$ and $0<\rho<r$ we define

$$S_b^{n-m-1}(\rho)=\{y\in\mathbb{R}^n:|y-b|=\rho,\ y'=\alpha(b'-a')+a',\ \alpha\in\mathbb{R}^1\}. \tag{4.2}$$

It can be shown as a direct consequence of the definition that these spheres are linked. Also, see [Gage81, introductory remark].

For any $a\in\mathbb{R}^n$, let $F_a:\mathbb{R}^n\to\mathbb{R}^{n-m}$ be defined as $F_a(x)=F_a(x',x'')=(|x'-a'|,x'')\in\mathbb{R}^{n-m}$. Thus, for $z=(z_1,\ldots,z_{n-m})\in\mathbb{R}^{n-m}$, we have

$$\begin{aligned} F_a^{-1}(z) &= \{y\in\mathbb{R}^n:|y'-a'|=z_1,\ y''=(z_2,\ldots,z_{n-m})\} \\ &= S^m_{(a',z_2,\ldots,z_{n-m})}(z_1). \end{aligned}$$

It is not difficult to verify that $JF_a=1$ and that F_a is Lipschitz with Lipschitz constant 1. Let $I_r\subset\mathbb{R}^{n-m}$ denote the cube in $\mathbb{R}^{n-m}$ of side length $r>0$ and center $(r,0,\ldots,0)$. Then $F_a^{-1}(I_r):=\bigcup_{w\in I_r}F_a^{-1}(w)$ defines a "rectangular torus". For example, if $n=3$, $m=1$, $a=0\in\mathbb{R}^3$, and I_r is the r by r square in the (y,z)-plane with center $(r,0)$, then $F_a^{-1}(I_r)$ is the figure obtained by rotating I_r about the z-axis.

Theorem 3. *Let $u\in W^{1,p}(\mathbb{R}^n)$, let m be an integer with $n\ge m+2\ge 3$, $p>m$ and let $\mathbf{u}$ denote an arbitrary, but fixed, p-quasicontinuous representative of u as determined by the pointwise limit of a suitable subsequence of smooth functions u_k that converge strongly to u in $W^{1,p}(\mathbb{R}^n)$. Then:*

(i) $\mathbf{u}$ is continuous on $F_a^{-1}(w)$ for H^{n-m}-a.e. $w\in\mathbb{R}^{n-m}$.

(ii) If $a\in\mathbb{R}^n$ is an L^p-Lebesgue point for both u and $|\nabla u|$ and if

$$\bar{u}(a)>0,$$

then there exists $R_0>0$ such that for every $0<r<R_0$ there exists an H^{n-m}-measurable set $E_r\subset I_r$ of positive H^{n-m}-measure such that $\mathbf{u}$ is continuous and positive on $F_a^{-1}(w)$ for $w\in E_r$.

Proof. (*i*) Since $u \in W^{1,p}(\mathbb{R}^n)$, we know that u is the strong limit of functions $u_k \in C^\infty(\mathbb{R}^n)$ and therefore, for each $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \mathbb{R}^n$ and a subsequence such that $\gamma_p(U_\varepsilon) < \varepsilon$ and that the u_k converge to u uniformly on $\mathbb{R}^n \setminus U_\varepsilon$; cf. [MZ97, Lemma 2.19]. Choosing a sequence $\varepsilon_j \to 0$, we see that $\gamma_p(U) = 0$ where $U := \bigcap_{\varepsilon_j} U_{\varepsilon_j}$. Since F_a is Lipschitz, $\gamma_p\left(F_a\left(U_{\varepsilon_j}\right)\right) \le C\gamma_p\left(U_{\varepsilon_j}\right) < C\varepsilon_j$, where $C = C(p,n)$, [AH96, Theorem 5.2.1]. Let

$$E := \bigcap_{\varepsilon_j > 0} F_a\left(U_{\varepsilon_j}\right).$$

Then $\gamma_p(E) = 0$, so that $H^{n-p+\varepsilon}(E) = 0$ for all $\varepsilon > 0$, by (2.1). Since $p > m$, there exists $\varepsilon > 0$ and $0 < \alpha < 1$ such that $n - p + \varepsilon = n - m - \alpha$, and therefore $H^{n-m-\alpha}(E) = 0$. This, in turn, implies that $H^{n-m}(E) = 0$. If $w \notin E$, then $w \notin F_a(U_{\varepsilon_j})$ for some j, which implies that $F_a^{-1}(w) \cap U_{\varepsilon_j} = \emptyset$. Thus $\mathbf{u}$, the uniform pointwise limit of the u_k on $\mathbb{R}^n \setminus U_{\varepsilon_j}$, is continuous on $F_a^{-1}(w)$ for $w \notin E$. That is, $\mathbf{u}$ is continuous on $F_a^{-1}(w)$ for H^{n-m}-a.e. $w \in \mathbb{R}^{n-m}$.

(*ii*) The proof is divided into three parts.

Step 1. For H^{n-m}-a.e. $w \in I_r$, $u_w := u \llcorner F_a^{-1}(w)$, we claim that

$$\sup_{F_a^{-1}(w)} |\mathbf{u}| \le C\left(\int_{F_a^{-1}(w)} r^{p-m}\left|\nabla(u_w)\right|^p + r^{-m}|u_w|^p \, dH^m\right)^{1/p}, \tag{4.3}$$

with C a constant. For this, observe that the co-area formula yields

$$\begin{aligned}\lim_{k\to\infty}&\int_{I_r}\int_{F_a^{-1}(w)} |\nabla u_k - \nabla u|^p + |u_k - u|^p \, dH^m \, dH^{n-m}(w)\\ &= \lim_{k\to\infty}\int_{F_a^{-1}(I_r)} \left(|\nabla u_k - \nabla u|^p + |u_k - u|^p\right) dH^n\\ &= 0.\end{aligned}$$

Thus there is a subsequence of the u_k (still denoted as the full sequence) such that for H^{n-m}-a.e. $w \in I_r$,

$$\lim_{k,l\to\infty}\int_{F_a^{-1}(w)} |\nabla u_k - \nabla u_l|^p + |u_k - u_l|^p \, dH^m = 0. \tag{4.4}$$

This subsequence converges strongly to some element of $W^{1,p}(F_a^{-1}(w))$, which we denote by

$$u \llcorner F_a^{-1}(w). \tag{4.5}$$

Since $u_k \to \mathbf{u}$ uniformly on $F_a^{-1}(w)$ for $w \notin E$, observe that $\mathbf{u} \llcorner F_a^{-1}(w)$ is a continuous representative of $u \llcorner F_a^{-1}(w)$. To ease notation, we will write u_w for $u \llcorner F_a^{-1}(w)$. For $g \in C^\infty(\mathbb{R}^n)$, it is well known that

$$\sup_{S_a^m(1)} |g| \le C\left(\int_{S_a^m(1)} |\nabla g|^p + |g|^p \, dH^m\right)^{1/p},$$

with $C = C(m,p)$, and by a simple scaling argument that

$$\sup_{S_a^m(r)} |g| \le C\left(\int_{S_a^m(r)} r^{p-m}|\nabla g|^p + r^{-m}|g|^p \, dH^m\right)^{1/p}. \tag{4.6}$$

Since $u_k \llcorner F_a^{-1}(w)$ converges uniformly to $\mathbf{u} \llcorner F_a^{-1}(w)$ and strongly to u_w in the sense of (4.4), applying (4.6) with g replaced by u_k yields (4.3).

Step 2. We will show that there exist a constant $C_2 > 0$ and an H^{n-m}-measurable set $E_r \subset I_r$ of positive H^{n-m}-measure such that

$$\int_{F_a^{-1}(w)} |\nabla u|^p + \left|\frac{u-\bar{u}(a,r)}{r}\right|^p dH^m \le C_2 r^m \text{ for each } w \in E_r. \tag{4.7}$$

From the hypotheses that a is an L^p-Lebesgue point for both u and $|\nabla u|$ and that $\bar{u}(a) > 0$, it follows that there exist positive numbers R and κ such that for $r \in (0, R)$ we have

$$\bar{u}(a,r) > \kappa > 0 \tag{4.8}$$

and

$$\int_{B_a^n(r)} |\nabla u|^p \, dH^n \le \left(|\overline{\nabla u}(a)|^p + 1\right) H^n(B_a^n(r)). \tag{4.9}$$

Using Poincarè's inequality and (4.9), there exists $C_1 = C_1(n,p)$ such that

$$\begin{aligned}\int_{B_a^n(r)} |u - \bar{u}(a,r)|^p \, dH^n &\le C_1 \int_{B_a^n(r)} |\nabla u|^p \, r^p dH^n \\ &\le C_1 \alpha_n \left(|\overline{\nabla u}(a)|^p + 1\right) r^{n+p}\end{aligned}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$, and consequently,

$$\int_{B_a^n(r)} \left|\frac{u-\bar{u}(a,r)}{r}\right|^p dH^n \le C_1\alpha_n \left(|\overline{\nabla u}(a)|^p + 1\right) r^n \text{ for } r \in (0,R). \tag{4.10}$$

Employing the co-area formula, (4.10) and (4.9), we have for all $r \in (0, R)$,

$$\begin{aligned}\int_{I_r} \int_{F_a^{-1}(w)} & |\nabla u|^p + \left|\frac{u-\bar{u}(a,r)}{r}\right|^p dH^m(t)\, dH^{n-m}(w) \\ &= \int_{F_a^{-1}(I_r)} |JF_a| \left(|\nabla u|^p + \left|\frac{u-\bar{u}(a,r)}{r}\right|^p\right) dH^n \\ &\le \int_{B_a^n\left(r + r\sqrt{n}/2\right)} \left(|\nabla u|^p + \left|\frac{u-\bar{u}(a,r)}{r}\right|^p\right) dH^n \\ &\le \alpha_n \left(1 + \frac{\sqrt{n}}{2}\right)^n (C_1+1)\left(|\overline{\nabla u}(a)|^p + 1\right) r^n.\end{aligned}$$

That is, setting $C_2 = \alpha_n \left(1 + \sqrt{n}/2\right)^n (C_1 + 1)\left(|\overline{\nabla u}(a)|^p + 1\right)$, we have

$$\int_{I_r} \int_{F_a^{-1}(w)} |\nabla u|^p + \left|\frac{u-\bar{u}(a,r)}{r}\right|^p dH^m \, dH^{n-m}(w) \le C_2 r^n.$$

Let $G(w)$ denote the inner integral in this expression, so that we have

$$\int_{I_r} G(w) \, dH^{n-m}(w) \le C_2 r^n,$$

which establishes (4.7).

Step 3. Finally, we will establish *(ii)* of our theorem. Since $E_r \subset I_r$, notice that for $w \in E_r$, $F_a^{-1}(w)$ is an m-sphere whose radius, $w_1 =: \rho$, has the property that $r/2 \le \rho \le 3r/2$. Thus, using (4.3) and (4.7), we obtain

$$\begin{aligned}
&\sup_{F_a^{-1}(w)} \left| \frac{\mathbf{u} - \bar{u}(a,r)}{r} \right|^p \\
&\quad \le C \int_{F_a^{-1}(w)} \left(\rho^{p-m} \left| \nabla \left(\frac{u_w - \bar{u}(a,r)}{r} \right) \right|^p + \rho^{-m} \left| \frac{u_w - \bar{u}(a,r)}{r} \right|^p \right) dH^m \\
&\quad \le C \left(\frac{3}{2} \right)^p \rho^{-m} \int_{F_a^{-1}(w)} \left(|\nabla (u_w)|^p + \left| \frac{u_w - \bar{u}(a,r)}{r} \right|^p \right) dH^m \\
&\quad \le C \left(\frac{3}{2} \right)^p 2^m r^{-m} \int_{F_a^{-1}(w)} \left(|\nabla (u_w)|^p + \left| \frac{u_w - \bar{u}(a,r)}{r} \right|^p \right) dH^m \\
&\quad \le C_2\, C \left(\frac{3}{2} \right)^p 2^m.
\end{aligned}$$

With $K := \left(C_2\, C \left(\frac{3}{2}\right)^p 2^m \right)^p$, we have $\sup_{F_a^{-1}(w)} |\mathbf{u} - \bar{u}(a,r)| \le Kr$ for $w \in E_r$. This, along with (4.8), implies there exists $R_0 > 0$ such that $\mathbf{u} > 0$ on $F_a{}^{-1}(w)$ for $w \in E_r,\ 0 < r < R_0$. □

Theorem 4. *Let $n \ge 3$, $n > m$ and $p > m \ge n - 1 - m \ge 1$. If u is a non-zero element of $W^{1,p}(\mathbb{R}^n)$, then u has a pair of linked spheres of dimensions m and $n - 1 - m$ in its support.*

Proof. If $u \in W^{1,p}(\mathbb{R}^n)$ is not identically zero, then there exists $a \in \mathbb{R}^n$ such that a is an L^p-Lebesgue point for u and $|\nabla u|$. We will assume without loss of generality, that $\bar{u}(a) > 0$. Applying Theorem 3 we obtain $r_0 > 0$ and a Borel set $E_{r_0} \subset I_{r_0}$ of positive H^{n-m}-measure such that for $w \in E_{r_0} \subset \mathbb{R}^{n-m}$, $\mathbf{u}$ is continuous and positive everywhere on $F_a^{-1}(w) = S^m_{(a', w_2, \ldots, w_{n-m})}(w_1)$. With a slight abuse of the notation introduced at the beginning of Section 4, we let $w'' := (w_2, \ldots, w_{n-m})$ so that we now have

$$F_a^{-1}(w) = S_{(a', w'')}(w_1).$$

Let $W_a := \bigcup_{w \in E_{r_0}} F_a^{-1}(w)$. Since E_{r_0} is H^n-measurable and $JF_a = 1$, we can appeal to the co-area formula to conclude that

$$H^n(W_a) = \int_{E_{r_0}} H^m\left(F_a^{-1}(w)\right)\, dH^{n-m}(w) > 0.$$

Note that $\mathbf{u}$ is defined and is positive at all points of W_a. For suitable $w \in W_a$, we will construct an $(n - m - 1)$-sphere that will link with $S^m_{(a', w'')}(w_1)$ and that will lie in a "radial" $(n - m)$-plane emanating from (a', w'') orthogonal to $Q^{-1}(w'')$. For this purpose define

$$P\colon \mathbb{R}^n \setminus \bar{B}^n_{(a', w'')}(r_0/2) \to S^m_{(a', w'')}(1) \quad \text{by } P(x) = \left(a' + \frac{x' - a'}{|x' - a'|}, w'' \right).$$

Observe that P is locally Lipschitz and that $P^{-1}(\theta)$ is independent of w for $\theta \in S^m_{(a', w'')}(1)$. Proceeding as in the proof of Theorem 3, Step 1, with F_a replaced by P, an application of the co-area formula yields that $u \llcorner P^{-1}(\theta) \in W^{1,p}(P^{-1}(\theta))$ for H^m-a.e. $\theta \in S^m_{(a', w'')}(1)$ and that $\mathbf{u} \llcorner P^{-1}(\theta)$ is a p-quasicontinuous representative for $u \llcorner P^{-1}(\theta)$; see (4.4) and (4.5). Since $H^n(W_a) > 0$, the co-area formula also implies that $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ for H^m-a.e $\theta \in S^m_{(a',w'')}(1)$. Thus, for such θ, there exists

$$w \in W_a \cap P^{-1}(\theta) \tag{4.11}$$

such that w is a Lebesgue point for both $u \llcorner P^{-1}(\theta)$ and $\nabla(u \llcorner P^{-1}(\theta))$. Since $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ and $\mathbf{u} \llcorner P^{-1}(\theta) > 0$ on $W_a \cap P^{-1}(\theta)$, it follows that we can also require w to have been chosen so that

$$\overline{u \llcorner P^{-1}(\theta)}(w) > 0. \tag{4.12}$$

With w determined by (4.11) and (4.12), it follows that $u \llcorner P^{-1}(\theta)$ satisfies the hypotheses of Theorem 3 (ii) with the ambient space $\mathbb{R}^n$ replaced by $P^{-1}(\theta)$ and with F_a replaced by $D\colon P^{-1}(\theta) \to \mathbb{R}^1$, defined by $D(x) = |x - (a', w'')|$. Theorem 3 (ii) provides a number $0 < \bar{r} < w_1/2$ and a set $A \subset (\bar{r}/2, 3\bar{r}/2)$ of positive H^1-measure such that $\mathbf{u} \llcorner P^{-1}(\theta)$ is defined and positive on each $D^{-1}(\rho)$, $\rho \in A$. Thus we have that $\mathbf{u} > 0$ on the $(n-m-1)$-sphere $D^{-1}(\rho)$ and $\mathbf{u} > 0$ on the m-sphere $S^m_{(a',w'')}(r)$. These spheres are linked, since they are similar to the linked spheres (4.1) and (4.2). □

Theorem 5. *Let $h\colon \bar{B}^k \to \mathbb{R}^n$ be an embedding of the closed ball $\bar{B}^k \subset \mathbb{R}^{k+1}$ where $1 \le k < n$ and $n \ge 3$. If $u \in W^{1,p}(\mathbb{R}^n)$, $p > [\frac{k+1}{2}]$ and $\operatorname{spt} u \subset h(\bar{B}^k)$, then $u \equiv 0$.*

Proof. First, assume k is even, $k+1 < n$, and by contradiction, suppose that $H^n(\operatorname{spt} u) > 0$. Writing $x \in \mathbb{R}^n$ as $x = (x', x'')$ where $x' \in \mathbb{R}^{k+1}$, recall that $Q\colon \mathbb{R}^n \to \mathbb{R}^{n-k-1}$ is defined as $Q(x) := x''$. Then we have $H^{k+1}(Q^{-1}(x'') \cap \operatorname{spt} u) > 0$ for all x'' in a set E of positive H^{n-k-1}-measure and, as in (4.5), u is a nonzero element of $W^{1,p}(Q^{-1}(x''))$ for H^{n-k-1}-a.e. $x'' \in E$. Redefine E to include only such x''. For $x'' \in E$, we employ Theorem 4 with $\mathbb{R}^n$ replaced by the $(k+1)$-dimensional affine space $Q^{-1}(x'')$ and m replaced by $k/2$ to conclude that $u \in W^{1,p}(Q^{-1}(x''))$ contains a pair of linked spheres, both of dimension $k/2$, in its support. Call these spheres S_1 and S_2. With h as in the statement of our theorem, let $H := h^{-1} \llcorner \big(Q^{-1}(x'') \cap h(\bar{B}^k)\big)$; so H is a homeomorphism of $\big(Q^{-1}(x'') \cap h(\bar{B}^k)\big)$ into $\mathbb{R}^k$. Since S_1 and S_2 are linked spheres in $\big(Q^{-1}(x'') \cap h(\bar{B}^k)\big)$ and since H is a homeomorphism, it follows from Definition 1 that $H(S_1)$ and $H(S_2)$ are linked in $\mathbb{R}^k$, which contradicts Theorem 2.

The above proof is easily modified and simpler for the case $k+1 = n$. A similar argument holds when k is odd. □

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