MASTER'S COMPREHENSIVE EXAMINATION
STATISTICAL INFERENCE
SPRING SEMESTER, 2007

<u>Directions</u>: Work 7 out of the following 10 problems.

1. Let $X_1, X_2, ..., X_n$ be a sample of independent Bernoulli trials with success probability *p*. Show that $X = \sum_{i=1}^{n} X_i$ has the Binomial(*n*, *p*) distribution.

2. A random variable *X* is said to have a Weibull distribution with parameters λ and θ if it has pdf $f(x) = \frac{\theta}{\lambda}(\frac{x}{\lambda})^{\theta-1} \exp[-(\frac{x}{\lambda})^{\theta}]$, $0 < x < \infty$, $\lambda > 0, \theta > 0$.
   a) Verify that $f(x)$ is a pdf.
   b) Let $X_1, X_2, ..., X_n$ be a random sample from the distribution above. Find the Maximum Likelihood estimator $\hat{\lambda}$ for λ. What is the distribution of $n[\hat{\lambda}]^{\theta}$?

3. Let $X_1, X_2, ..., X_n$ be a random sample from the exponential distribution with parameter θ.
   a) Find the Maximum Likelihood estimator for θ.
   b) Use your result in part a) to find the MLE for $e^{5\theta}$, and derive its asymptotic distribution using asymptotic maximum likelihood theory.
   c) Use the central limit theorem to derive the result in b)

4. Let $X_1, X_2, ..., X_n$ be a random sample from $N(0, \sigma^2)$
   a) Find the UMVUE for σ^2.
   b) Show that your answer to part a) is statistically independent to $\frac{X_{(1)}}{X_{(2)}}$, the ratio of the first two order statistics. (*Hint*: You shouldn't have to work too hard here.)

5. Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution with parameter *p*. If *p* is restricted so that we know that $\frac{1}{2} \le p \le 1$, find the MLE of this parameter.

6. Let $X_1, X_2, ..., X_n$ be a random sample from $Gamma(4,\theta)$.
   a) Use the likelihood ratio to write the asymptotic α = 0.05 critical region for the test $Ho: \theta = 7$ vs $Ha: \theta \neq 7$. Indicate the test statistic on which the test is based.
   b) What is the uniformly most powerful test for testing $Ho: \theta = 7$ vs $Ha: \theta > 7$?
   c) Write (but do not evaluate) an integral expression for the power for the case when $\theta = 10$.

7. Let $X_1, X_2, ..., X_n$ i.i.d. $N(\mu_1, \sigma^2 = 10)$ and $Y_1, Y_2, ..., Y_m$ i.i.d. $N(\mu_2, \sigma^2 = 10)$.
   a) Perform the Likelihood ratio test for

      Ho: $\mu_1 = \mu_2$

      Ha: $\mu_1 \neq \mu_2$ with α = 0.05.
   b) Based on the sample of size $n = 20$, $\bar{x} = 12.5$ and $m = 30$ provides $\bar{y} = 10.2$.

      Do the approximate test using χ^2-distribution.

8. Let X_1 and X_2 be random variables from Exponential distribution with a mean *β*.
   a) Find the joint density function of $U = X_1 + X_2$ and $V = \frac{X_1}{X_2}$
   b) Are random variables *U* and *V* independent?

9. Suppose $X_1, X_2, ..., X_n$ be an independent random variable with $X_i \sim Poisson(\frac{\theta}{i})$, *i*=1,2,…,*n*. The parameter *θ* is unknown and the parameter space is $\Theta = (0, \infty)$. Define $c_n = \frac{1}{n}\sum_{i=1}^{n} i^{-1}$.
   a) Identify a sufficient statistic for *θ* and prove that it is sufficient using the factorization theorem.
   b) The maximum likelihood estimator of *θ* is $\frac{\bar{X}}{c_n}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Calculate the mean squared error of the MLE of *θ*.

10. Let $X_{1:n}, X_{2:n} < ... < X_{n:n}$ be the order statistics of a random sample of size *n* from a distribution with pdf $f(x) = 1, 0 < x < 1$, zero elsewhere. Show that the *k*th order statistic $X_{k:n}$ has a beta pdf with parameters *α* = *k* and *β* = *n* – *k* + 1.