MASTER'S COMPREHENSIVE EXAMINATION
STATISTICAL INFERENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
SPRING SEMESTER, 2006

Directions: Work 7 out of the following 10 problems.

1. Let X and Y have the joint pdf $f_{X,Y}(x,y) = 2e^{-x-y}$, $0 < x < y < \infty$. Define $U = 2X$ and $V = Y - X$.
   a) Find the joint probability density function of U and V.
   b) Are U and V independent?. Show your work.

2. Let $X_1, X_2, ..., X_n$ denote a random sample from Beta (θ, 1) distribution.
   a) Find the Maximum Likelihood Estimator of θ, $\hat{\theta}_{MLE}$.
   b) Find the Rao-Cramer lower bound.
   c) Discuss the efficiency of $\hat{\theta}_{MLE}$.

3. Given the pdf of t – distribution with degrees of freedom n is $f(t) = \dfrac{\Gamma((n+1)/2)}{\sqrt{\pi n}\,\Gamma(n/2)} \dfrac{1}{(1+t^2/n)^{(n+1)/2}}$, show that it converges to the pdf of standard normal distribution.

4. Let S^2 be the sample variance of a random sample of size $n > 1$ from $N(\mu, \theta)$, $\theta > 0$, where μ is known. We know $E(S^2) = \theta$.
   a) What is the efficiency of S^2?
   b) Under these conditions, what is the $\hat{\theta}_{MLE}$?
   c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$?

5. Let $X_{1:5} < X_{2:5} < ... < X_{5:5}$ be the order statistics of a random sample of size $n = 5$ from a distribution with pdf $f_X(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$, $-\infty < x < \infty$, for all real θ. Find the Likelihood ratio test for testing Ho: $H_o : \theta = \theta_0$ $H_a : \theta \neq \theta_0$.

6. Let X have a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$.
   a) Find the Fisher information $I(\theta)$.
   b) If $X_1, X_2, ..., X_n$ is a random sample from this distribution, show that the $\hat{\theta}_{MLE}$ is an efficient estimator of θ.
   c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$?

7. Let $X_1, X_2, ..., X_n$ be a random sample from $f_X(x) = \theta e^{-\theta x}$, $x > 0$
   a) Find the sufficient statistic for θ.
   b) Find the Maximum Likelihood Estimator of θ.
   c) Find the Method of Moment Estimator of θ.
   d) Find the Minimum Variance Unbiased Estimator of θ.

8. Let $X_{1:4} < X_{2:4} < ... < X_{4:4}$ denote the order statistics of a random sample of size 4 from the Uniform(0, θ). Let the observed value of $X_{4:4}$ be x_4. We reject Ho: $\theta = 1$ and accept Ha: $\theta \neq 1$ if either $x_4 > 1$ or $x_4 \leq 1/2$. Find the power function $\gamma(\theta)$ of the test, $\theta > 0$.

9. Let $X_1, X_2, ..., X_n$ be a random sample from $f_X(x) = \frac{1}{3\theta}$, $-\theta < x < 2\theta$, where $\theta > 0$.
   a) Find the Maximum Likelihood Estimator of θ, $\hat{\theta}_{MLE}$.
   b) Is $\hat{\theta}_{MLE}$ a sufficient statistic for θ? Why?
   c) Is $(n+1)\hat{\theta}_{MLE}/n$ the unique MVUE (Minimum Variance Unbiased Estimator) of θ? Why?

10. Let $X_1, X_2, ..., X_n$ denote a random sample from a gamma distribution with $\alpha = 2$ and $\beta = \theta$. Let Ho: $\theta = 1$ and Ha: $\theta > 1$.
    a) Show that there exists a uniformly most powerful test for Ho against Ha, determine the statistic Y upon which the test may be based, and indicate the nature of the best critical region.
    b) Find the pdf of the statistic Y in part (a). If we want a significant level of 0.05, write an equation which can be used to determine the critical region. Let $\gamma(\theta)$, $\theta \geq 1$, be the power function of the test. Express the power function as an integral.