MASTER'S COMPREHENSIVE EXAMINATION
STATISTICAL INFERENCE
FALL SEMESTER, 2006

<u>Directions</u>: Work 7 out of the following 10 problems.

1. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution that has the pdf $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} I_{(X>0)}$. Find the Maximum Likelihood Estimator(MLE) and Minimum Variance Unbiased Estimator(MVUE) of $P(X \le 2)$.

2. Let X_1 and X_2 have the joint pdf $h(x_1, x_2) = 8x_1x_2$, $0 < x_1 < x_2 < 1$, zero elsewhere. Find the joint pdf of $Y_1 = X_1 / X_2$ and $Y_2 = X_2$.

3. Let $f(x)$ and $F(x)$ denote, respectively, the pdf and the cdf of the random variable *X*. The conditional pdf of *X*, given $X > x_0$, x_0 a fixed number, is defined by $f(x \mid X > x_0) = f(x)/[1 - F(x_0)]$, $x_0 < x$, zero elsewhere. This kind of conditional pdf finds application in a problem of time until death, given survival until time x_0.
   a) Show that $f(x \mid X > x_0)$ is a pdf.
   b) Let $f(x) = e^{-x} I_{(X>0)}$, and zero elsewhere. Compute $P(X > 2 \mid X > 1)$.

4. Let $X_{1:2} < X_{2:2}$ denote the order statistics of a random sample of size 2 from $N(0, \sigma^2)$.
   a) Show that $E(X_{1:2}) = -\sigma / \sqrt{\pi}$
   b) Find the covariance of $X_{1:2}$ and $X_{2:2}$.

5. Discuss the problem of finding a confidence interval for the difference $\mu_1 - \mu_2$ between the two means of two normal distributions if the variances σ_1^2 and σ_2^2 are known but not necessarily equal.

6. Let $X_1, X_2, ..., X_n$ be a random sample from $N(0, \theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum_{i=1}^{n} | X_i |$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

7. If $az^2 + bz + c = 0$ for more than two values of z, then $a = b = c = 0$. Use this result to show that the family {Binomial(2, θ) : 0 < θ < 1} is complete.

8. Let us say the life of a tire in miles, say X, is normally distributed with mean θ and standard deviation 5000. Past experience indicates that θ = 30,000. The manufacturer claims that the tires made by a new process have mean θ > 30,000. It is possible that θ = 35,000. Check his claim by testing Ho: θ = 30,000 against Ha: θ > 30,000. We shall observe n independent values of X, say $x_1, x_2, ..., x_n$, and we shall reject Ho if and only if $\bar{x} \geq c$. Determine n and c so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30{,}000)$ =0.01 and $\gamma(35{,}000) = 0.98$.

9. Let X be a positive random variable; i.e., P($X \leq 0) = 0$. Argue that
   a) $E(1/X) \geq 1/E(X)$,
   b) $E(-\log X) \geq -\log[E(X)]$,
   c) $E(X^3) \geq [E(X)]^3$.

10. Consider a random sample $X_1, X_2, ..., X_n$ from a distribution with pdf $f(x;\theta) = \theta(1-x)^{\theta-1}, 0 < x < 1$, zero elsewhere, where θ >0.
    a) Find the form of the uniformly most powerful test of Ho: θ = 1 against Ha: θ > 1.
    b) What is the likelihood ratio for testing Ho: θ = 1 against Ha: θ ≠ 1?