

Bilinear Forms on Frobenius Algebras

Will Murray

*Department of Mathematics and Statistics
California State University, Long Beach
1250 Bellflower Boulevard
Long Beach, CA, 90840-1001*

Abstract

We analyze the homothety types of associative bilinear forms that can occur on a Hopf algebra or on a local Frobenius k -algebra R with residue field k . If R is symmetric, then there exists a unique form on R up to homothety iff R is commutative. If R is Frobenius, then we introduce a norm based on the Nakayama automorphism of R . We show that if two forms on R are homothetic, then the norm of the unit separating them is central, and we conjecture the converse. We show that if the dimension of R is even, then the determinant of a form on R , taken in \dot{k}/\dot{k}^2 , is an invariant for R .

Key words: bilinear form, Frobenius algebra, homothety, Hopf algebra, isometry, local algebra, Nakayama automorphism, Ore extension, symmetric algebra

1 Introduction

Let R be a finite-dimensional algebra over a field k . We say R is a *Frobenius algebra* if there exists a nondegenerate bilinear form $B : R \times R \rightarrow k$ that is *associative* in the sense that $B(rs, t) = B(r, st) \forall r, s, t \in R$. We say R is a *symmetric algebra* if there exists a nondegenerate associative symmetric bilinear form $B : R \times R \rightarrow k$. These properties are equivalent to the existence of an isomorphism between R and its k -dual $\hat{R} := \text{Hom}_k(R, k)$ as left R -modules, respectively, as (R, R) -bimodules.

Email address: wmurray@csulb.edu (Will Murray).

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Since many different isomorphisms between R and \hat{R} exist, we also have many bilinear forms. A natural question to ask is whether the various forms are *isometric*, that is, the same under change of basis. It is trivial to observe that any form may be scaled by a nonzero constant from k , so we define two forms B and B' to be *homothetic* if there exists a change of basis $V \in \text{Aut}_k(R)$ and a scalar $\alpha \in \dot{k} := k - \{0\}$ such that $B'(r, s) = \alpha B(Vr, Vs) \forall r, s \in R$. We then ask instead when two forms on R are homothetic.

In this paper we will study the question above in the case when R has an ideal \mathfrak{m} with $R/\mathfrak{m} \simeq k$. For example, this condition is satisfied by the group algebra $R = kG$, where G is any finite group, by taking \mathfrak{m} to be the kernel of the augmentation map $\epsilon(\sum \alpha_g g) = \sum \alpha_g \in k$. It is also true for Hopf algebras, whose definition includes the existence of a counit $\epsilon : R \rightarrow k$. For most of our results we will also need to assume that k has good characteristic.

We will show that in the local symmetric case, there exists a unique form on R up to homothety iff R is commutative. For Frobenius algebras that are not symmetric, we will introduce a norm based on the order of the Nakayama automorphism, a distinguished k -algebra automorphism of R that measures how far R is from being a symmetric algebra. (The automorphism is the identity iff R is symmetric.) We will show that if two forms on R are homothetic, then the norm of the unit separating them is central, and we will conjecture the converse. Finally, we will study the determinant of a form on R and show that in even dimension, the value of the determinant in \dot{k}/\dot{k}^2 is an invariant of the algebra.

The idea of comparing an algebra with its dual was pioneered by F. G. Frobenius himself ([2]) in connection with representations of finite groups, and group algebras have remained important examples of symmetric algebras. Nakayama gave new examples and developed the main properties of Frobenius algebras and symmetric algebras in [7–9]. More recently, the group algebra example was generalized when Larson and Sweedler ([5]) showed that all finite dimensional Hopf algebras are Frobenius. (See [1] for a treatment of the ubiquity of Hopf algebras.) Modern interest in Frobenius algebras has grown far beyond their representation-theoretic origins as connections have been discovered to such diverse areas as topological quantum field theories, Gorenstein rings in commutative algebra, coding theory, and the Yang-Baxter Equation. For an excellent reference on the subject, see [3].

2 Preliminaries and examples

Let k be a field and R be a finite-dimensional k -algebra. Throughout this paper, we will use the word *form* (respectively, *symmetric form*) to mean

a nondegenerate bilinear form (respectively, nondegenerate symmetric bilinear form) $B : R \times R \rightarrow k$ that is *associative* in the sense that $B(rs, t) = B(r, st) \forall r, s, t \in R$.

In [3], Theorems 3.15 and 16.54, we have:

Theorem 1 *The following conditions are equivalent:*

1. $R \simeq \hat{R}$ as left R -modules (respectively, as (R, R) -bimodules).
2. There exists a linear functional $\lambda : R \rightarrow k$ whose kernel contains no nonzero left ideals. (Respectively, $\lambda(rs) = \lambda(sr) \forall r, s \in R$.)
3. There exists a form (respectively, symmetric form) $B : R \times R \rightarrow k$.

If R satisfies these conditions, R is said to be a Frobenius (respectively, symmetric) algebra. \square

For any finite-dimensional algebra R , the dual \hat{R} is isomorphic as a left R -module to the injective hull of $R/\text{rad}R$, so the isomorphism type of ${}_R\hat{R}$ does not depend on the ground field k . In ([6]) it is shown that the isomorphism type of \hat{R} as an (R, R) -bimodule is also independent of k ; thus we may speak of R being a Frobenius (respectively, symmetric) algebra without reference to the ground field.

The equivalence of the first two conditions in Theorem 1 follows from taking λ to be the image of 1 under the module isomorphism and vice versa. The equivalence of the last two follows from defining $B(r, s) := \lambda(rs)$ and $\lambda(r) := B(r, 1)$. Since the third condition is right-left symmetric, we could also include the right-handed analogues of the other conditions above.

Given one isomorphism $\varphi : {}_R R \xrightarrow{\sim} {}_R \hat{R}$, any other isomorphism φ' is obtained by composition with an automorphism of the left regular module ${}_R R$, which corresponds to right multiplication by a unit $u \in U(R)$. This affects the other conditions above as follows: the new functional is $\lambda' = u\lambda : r \mapsto \lambda(ru)$; and the new form is $B'(r, s) = B(r, su)$.

If the conditions of Theorem 1 hold, the nondegeneracy of the form B implies that there is a unique k -linear map $\sigma : R \rightarrow R$ defined by $B(r, s) = B(s, \sigma(r)) \forall r, s \in R$. (Equivalently, $\lambda(rs) = \lambda(s\sigma(r)) \forall r, s \in R$.) It is easy to check that σ is actually a k -algebra automorphism of R , known as the *Nakayama automorphism* of R . Replacing B with a new form B' defined by the unit u gives us the new automorphism $\sigma' = I_u \circ \sigma$, where I_u is the inner automorphism $r \mapsto uru^{-1}$. So the Nakayama automorphism is determined up to composition with inner automorphisms; equivalently, it is a well-defined element of the group of outer automorphisms of R . The algebra is symmetric iff σ can be taken to be the identity, iff the Nakayama automorphism determined by an arbitrary form is an inner automorphism.

In most theorems we will assume that R has an ideal \mathfrak{m} with $R/\mathfrak{m} \simeq k$. As shown in the previous section, this condition is satisfied when R is a finite-dimensional Hopf algebra or when R is the group algebra of any finite group over any field.

We will sometimes assume additionally that R is local, or, equivalently, that $\mathfrak{m} = \text{rad } R$. This condition is satisfied, for example, by the group algebra kG where G is a finite p -group and $\text{char } k = p$.

Another classic example that satisfies both assumptions above is the Nakayama-Nesbitt example. This algebra is described in [7] in terms of matrices, but we will use polynomials to save space. Let α be a fixed element of k . We define

$$R_\alpha := k\langle x, y \rangle / (x^2 = y^2 = 0, yx = \alpha xy) = k \oplus kx \oplus ky \oplus kxy,$$

a four-dimensional local algebra with maximal ideal $\mathfrak{m} = (x, y)$. The functional $\lambda(a + bx + cy + dxy) = d$ shows that R_α is Frobenius. The Nakayama automorphism is $\sigma : x \mapsto \alpha^{-1}x, y \mapsto \alpha y$, since, for example, $\lambda(x\sigma(y)) = \lambda(yx) = \lambda(x(\alpha y))$. The algebra is symmetric iff $\alpha = 1$, and the Nakayama automorphism has finite order iff α is a root of unity in k .

In Example 17 we will extend this example by increasing the index of nilpotency of \mathfrak{m} . It is also possible to increase the number of generators of \mathfrak{m} , in which case the relations can be arranged so that the Nakayama automorphism effects any desired permutation among the generators.

3 The transpose of an endomorphism

Let R be a Frobenius algebra with form B . Given a k -vector space endomorphism $\varphi \in \text{End}(R_k)$, the nondegeneracy of B gives us a unique endomorphism φ^t , the *transpose* of φ with respect to B , that satisfies

$$B(\varphi r, s) = B(r, \varphi^t s) \quad (\forall r, s \in R).$$

We have the standard properties:

$$\begin{aligned} (\varphi + \psi)^t &= \varphi^t + \psi^t, \\ (\varphi\psi)^t &= \psi^t\varphi^t, \\ (\alpha\varphi)^t &= \alpha(\varphi^t) \quad (\forall \alpha \in k), \\ (\varphi^{-1})^t &= (\varphi^t)^{-1} \text{ if } \varphi \text{ is invertible.} \end{aligned}$$

In general, however, it is not true that $(\varphi^t)^t = \varphi$, unless B is symmetric. We

will see that it depends on the order of the Nakayama automorphism σ . We will use the notation $\varphi^{t^2} := (\varphi^t)^t$, and so on.

Lemma 2 $\varphi^{t^2} = \sigma\varphi\sigma^{-1}$.

PROOF. For all $r, s \in R$,

$$B(r, \varphi^{t^2}s) = B(\varphi^t r, s) = B(\sigma^{-1}s, \varphi^t r) = B(\varphi\sigma^{-1}s, r) = B(r, \sigma\varphi\sigma^{-1}s). \quad \square$$

Corollary 3 *If $\sigma^n = \text{Id}$, then $\varphi^{t^{2n}} = \varphi$.* \square

Now for any $x \in R$, let $\ell_x, \rho_x \in \text{End}_k(R)$ be left and right multiplication by x , respectively.

Lemma 4 *For all $i \geq 0$,*

- $\rho_x^{t^{2i}} = \rho_{\sigma^i(x)}$.
- $\rho_x^{t^{2i+1}} = \ell_{\sigma^i(x)}$.

PROOF. For all $r, s \in R$,

$$\begin{aligned} B(r, \rho_x^t s) &= B(\rho_x r, s) \\ &= B(rx, s) \\ &= B(r, xs) \\ &= B(r, \ell_x s), \end{aligned}$$

proving the second statement for $i = 0$. Similarly, for all $r, s \in R$,

$$\begin{aligned} B(r, \rho_x^{t^2} s) &= B(\rho_x^t r, s) \\ &= B(xr, s) \text{ by the above} \\ &= B(x, rs) \\ &= B(rs, \sigma(x)) \\ &= B(r, s\sigma(x)) \\ &= B(r, \rho_{\sigma(x)} s), \end{aligned}$$

proving the first statement for $i = 1$. The general statements follow by induction. \square

Corollary 5 *Even “transpose powers” of ρ_x commute with odd “transpose powers” of ρ_x .* \square

Now let B' be another form on R such that $B'(r, s) = B(r, su) \forall r, s \in R$, where $u \in U(R)$. The transpose gives us a way to determine when the two forms are homothetic:

Lemma 6 *B and B' are homothetic iff $\exists \alpha \in \dot{k}$ and $V \in \text{Aut}_k(R)$ such that $\rho_u = \alpha V^t V \in \text{Aut}_k(R)$.*

PROOF. If $\rho_u = \alpha V^t V$, then for all $r, s \in R$,

$$B'(r, s) = B(r, su) = B(r, \rho_u s) = B(r, \alpha V^t V s) = \alpha B(Vr, Vs),$$

so B and B' are homothetic. The converse is similar. \square

4 Forms on local symmetric algebras

Throughout this section we will assume that $\text{char } k \neq 2$ and that R is a local symmetric algebra with maximal ideal \mathfrak{m} such that $R/\mathfrak{m} \simeq k$. This condition is satisfied, for example, by the group algebra kG where G is a finite p -group and $\text{char } k = p > 2$, or by the Nakayama-Nesbitt example above. We will show that up to homothety, R has a unique symmetric form, and it has a unique form iff it is commutative.

Some of the results here are similar to those derived by Watanabe in [10]. Watanabe studied the three-fold multilinear form θ on a symmetric algebra defined by $\theta(r, s, t) := \lambda(rst)$, where λ is as in Theorem 1. He showed (by elementary techniques) that algebras with isometric three-fold forms are isomorphic. (This is done without using our assumptions that the algebra be local or that $\text{char } k \neq 2$.) Watanabe also studied bilinear forms and is responsible for Lemma 7 below.

Throughout this section, let B and B' be two forms on R such that $B'(r, s) = B(r, su) \forall r, s \in R$, where $u \in U(R)$. As above, let $\rho_u, \ell_u \in \text{Aut}_k(R)$ be right and left multiplication by u .

Lemma 7 (*Lemma 2 in [10]*) *If B is symmetric, then B' is symmetric iff $u \in Z(R)$.*

PROOF. For all $r, s \in R$, we have

$$\begin{aligned}
B'(r, s) = B'(s, r) &\iff B(r, su) = B(s, ru) \\
&\iff B(r, su) = B(ru, s) \\
&\iff B(r, su) = B(r, us) \\
&\iff B(r, \rho_u s) = B(r, \ell_u s).
\end{aligned}$$

Using the nondegeneracy of B , we have that B' is symmetric iff $\rho_u = \ell_u$, iff $u \in Z(R)$. \square

Lemma 8 *Suppose $\text{char } k \neq 2$ and (R, \mathfrak{m}) is a local symmetric k -algebra with $R/\mathfrak{m} \simeq k$. If B is symmetric, then B and B' are homothetic iff $u \in Z(R)$.*

PROOF. If B and B' are homothetic, then B' is symmetric, so by Lemma 7, $u \in Z(R)$.

Conversely, suppose that $u \in Z(R)$; we claim that we can find $\alpha \in \dot{k}$ and $v \in Z(R)$ such that $\alpha v^2 = u$. If we can find such a v (necessarily in $U(R)$ since $u \in U(R)$), then we have

$$\rho_u = \alpha(\rho_v)^2 = \alpha \ell_v \rho_v = \alpha \rho_v^t \rho_v,$$

so B and B' are homothetic by Lemma 6.

To prove the claim, suppose that $u = \alpha + m \in R$ where $\alpha \in \dot{k}$ and $m \in \mathfrak{m}$. We set $v := 1 + a_1 m + a_2 m^2 + \dots$, where $a_i \in k$, and solve inductively for the a_i to satisfy

$$\begin{aligned}
&\alpha v^2 = u \\
\alpha + 2\alpha a_1 m + \alpha(2a_2 + a_1^2)m^2 + \dots &= \alpha + m.
\end{aligned}$$

(Note that $m^n = 0$ for some n , so the expression above will terminate.) So it suffices to solve

$$\begin{aligned}
2\alpha a_1 &= 1 \\
(2a_2 + a_1^2) &= 0 \\
(2a_3 + 2a_1 a_2) &= 0 \\
&\vdots
\end{aligned}$$

Since $2\alpha \in \dot{k}$, these equations can always be solved, proving the claim. \square

It is now easy to prove our main results on symmetric algebras.

Theorem 9 *Suppose $\text{char } k \neq 2$ and (R, \mathfrak{m}) is a local symmetric k -algebra with $R/\mathfrak{m} \simeq k$. Then there exists a unique symmetric form on R up to homothety.*

PROOF. Suppose B and B' are both symmetric forms on R . Then by Lemma 7, $u \in Z(R)$, so by Lemma 8, B and B' are homothetic. \square

Theorem 10 *Suppose $\text{char } k \neq 2$ and (R, \mathfrak{m}) is a local symmetric k -algebra with $R/\mathfrak{m} \simeq k$. Then there exists a unique homothety class of forms on R iff R is commutative.*

PROOF. From Lemma 8, the forms B and B' are homothetic iff $u \in Z(R)$.

If R is commutative, then certainly $u \in Z(R) \forall u \in U(R)$, so all forms are homothetic. (Alternatively, we could prove this implication by noting that any form on a commutative algebra is symmetric, so Theorem 9 shows that all forms are homothetic.)

Conversely, if all forms are homothetic, then $u \in Z(R) \forall u \in U(R)$. This implies that R is commutative, since (using the fact that R is local) for any $r \in R$, one of r or $1 + r$ is a unit, hence central, so r is central. \square

We remark that Frobenius algebras that are not symmetric may have a unique homothety class of forms without being commutative. In fact, the noncommutative Nakayama-Nesbitt algebra above does have a unique homothety class of forms. So we cannot omit the symmetric condition from the theorem above.

Theorem 10 is a special case of Theorem 15 and Conjecture 16, which treat the case in which the Nakayama automorphism has finite order instead of being the identity.

5 Forms on Frobenius algebras

We will try to generalize the results of the previous section to Frobenius algebras. We would like to focus on the case in which the Nakayama automorphism of R has finite order. However, since the Nakayama automorphism is only well-defined up to inner automorphism, we first examine the case when it has finite inner order, i.e. it has finite order as a member of the group of outer automorphisms, i.e. some power of any particular Nakayama automorphism is inner. In this case we will define a norm on R and use it to show that in good characteristic we can then find a new form whose Nakayama automorphism

actually has finite order. Then we will show that a necessary (and perhaps sufficient) condition for any other form to be homothetic to that form is that the norm of the associated unit be central.

Suppose R is a Frobenius k -algebra with form B and corresponding Nakayama automorphism σ . Recall from Section 2 that if $B'(r, s) = B(r, su)$, then B' has Nakayama automorphism $\sigma' = I_u \circ \sigma$, where I_u denotes the inner automorphism $r \mapsto uru^{-1}$. Hence, the inner order of the Nakayama automorphism is independent of the choice of the form. We use this to define our norm function on R : suppose σ has finite inner order n . Then for $r \in R$, we define $N_\sigma(r) := r\sigma(r) \cdots \sigma^{n-1}(r)$. Before using this norm to find another Nakayama automorphism that has finite order, we need some technical lemmas.

Lemma 11 *Suppose $\sigma^n = I_a$. Then $\sigma(a) = a$.*

PROOF. For all $r \in R$,

$$\begin{aligned} B(r, a) &= B(a, \sigma(r)) = B(\sigma(r), \sigma(a)) = \cdots \\ &\cdots = B(\sigma^n(r), \sigma^n(a)) = B(ar a^{-1}, a) = B(ar, 1) = B(a, r). \quad \square \end{aligned}$$

Lemma 12 *Suppose $\sigma^n = I_a$. Then $(\sigma')^n = I_{N_\sigma(u)a}$.*

PROOF. Let $r \in R$. Then

$$\begin{aligned} \sigma'(r) &= I_u \circ \sigma(r) = u\sigma(r)u^{-1} \\ (\sigma')^2(r) &= u\sigma(u)\sigma^2(r)\sigma(u^{-1})u^{-1} \\ &\vdots \\ (\sigma')^n(r) &= u\sigma(u) \cdots \sigma^{n-1}(u)\sigma^n(r)\sigma^{n-1}(u^{-1}) \cdots \sigma(u^{-1})u^{-1} \\ &= N_\sigma(u)I_a(r)(N_\sigma(u))^{-1} \\ &= I_{N_\sigma(u)} \circ I_a(r) \\ &= I_{N_\sigma(u)a}(r). \quad \square \end{aligned}$$

Lemma 13 *Suppose (R, \mathfrak{m}) is local Frobenius with $R/\mathfrak{m} \simeq k$. Suppose $\sigma^n = I_a$ and $\text{char } k \nmid n$. If $x \in U(R)$ satisfies $\sigma(x) = x$ and $\bar{x} \in \dot{k}^n$ in R/\mathfrak{m} , then there exists $u \in U(R)$ such that $\sigma(u) = u$ and $x = u^n$, so $N_\sigma(u) = x$.*

PROOF. Let $x = \alpha + m$ with $\alpha \in \dot{k}$, $m \in \mathfrak{m}$, and say $\alpha = u_0^n$ for some $u_0 \in \dot{k}$. We set $u = u_0 + u_1m + u_2m^2 + \cdots$ (this terminates since m is nilpotent) and

solve inductively for $u_i \in k$ to satisfy $u^n = x$, that is,

$$u_0^n + nu_0^{n-1}u_1m + \left[nu_0^{n-1}u_2 + \binom{n}{2}u_0^{n-2}u_1^2 \right] m^2 + \cdots = \alpha + m.$$

It suffices to solve

$$\begin{aligned} u_0^n &= \alpha \\ nu_0^{n-1}u_1 &= 1 \\ nu_0^{n-1}u_2 + \binom{n}{2}u_0^{n-2}u_1^2 &= 0 \\ &\vdots \end{aligned}$$

Since $n \neq 0$ in k , these equations can be solved for the u_i . Then $u^n = x$ and $u \in U(R)$ since $u_0 \neq 0$. Finally, since $\sigma(x) = x$ and $\sigma = \text{Id}$ on $\alpha \in k$, we have $\sigma(m) = m$, so $\sigma(u) = u$ and hence $N_\sigma(u) = u^n = x$. \square

Theorem 14 *Suppose (R, \mathfrak{m}) is local Frobenius with $R/\mathfrak{m} \simeq k$. Suppose $\sigma^n = I_a$ and $\text{char } k \nmid n$. Then there exists a form B' on R whose Nakayama automorphism σ' satisfies $(\sigma')^n = \text{Id}$.*

PROOF. Say $a = \alpha + m$ with $\alpha \in k, m \in \mathfrak{m}$. We may assume that $\alpha = 1$, since scalar multiples of a will yield the same inner automorphism. Now by Lemma 11, $\sigma(a) = a$, so $\sigma(a^{-1}) = a^{-1}$. Since $a^{-1} = 1 - m + m^2 - \cdots$ and $1 \in k^n$, Lemma 13 tells us that $a^{-1} = N_\sigma(u)$ for some $u \in U(R)$. Let B' be the form defined by $B'(r, s) = B(r, su)$ for $r, s \in R$; then by Lemma 12,

$$(\sigma')^n = I_{N_\sigma(u)a} = I_{a^{-1}a} = \text{Id}. \quad \square$$

Thus, in good characteristic we can find a form whose automorphism has finite order.

We remark in passing that in the Ore ring of right twisted polynomials $R[t, \sigma]$ (where $tr = \sigma(r)t \forall r \in R$), evaluation of the monomial t^i at $r \in R$ is defined as $N_i(r) := r\sigma(r) \cdots \sigma^{i-1}(r)$. (See [4].) Thus our norm $N_\sigma(r)$ coincides with evaluation of t^n at r .

Returning to Frobenius algebras, we have established that in many cases we can find a form whose Nakayama automorphism has finite order. We now ask when other forms will be homothetic to that form. We will show that a necessary (and perhaps sufficient) condition for this to occur is that the unit separating the two forms have a central norm.

As before, let B and B' be two forms on R such that $B'(r, s) = B(r, su)$, where $u \in U(R)$. Let B have Nakayama automorphism σ , and suppose σ has finite order n . We saw in Lemma 8 that if $n = 1$ (so R is symmetric, and $N_\sigma(u) = u$) then B' is homothetic to B iff $u \in Z(R)$. We now generalize that result. We can relax the condition that R be local and assume only that $R/\mathfrak{m} \simeq k$. This assumption is satisfied by the group algebra of any finite group over any field and by any finite-dimensional Hopf algebra.

Theorem 15 *Suppose R is Frobenius with an ideal \mathfrak{m} such that $R/\mathfrak{m} \simeq k$, and suppose $\sigma^n = \text{Id}$. If B and B' are homothetic, then $N_\sigma(u) \in Z(R)$.*

Conjecture 16 *The converse is true, too, i.e. if R is Frobenius with an ideal \mathfrak{m} such that $R/\mathfrak{m} \simeq k$ and $\sigma^n = \text{Id}$, then $N_\sigma(u) \in Z(R)$ implies that B and B' are homothetic.*

Experimental evidence supports this conjecture, at least in the local case.

PROOF OF THEOREM 15 Let $\rho_u \in \text{Aut}_k(R)$ be right multiplication by u . If B and B' are homothetic, then by Lemma 6, $\exists \alpha \in k, V \in \text{Aut}_k(R)$ such that $\rho_u = \alpha V^t V \in \text{Aut}_k(R)$. Then for $i \geq 0$, we have

$$\begin{aligned}\rho_u^{t^{2i}} &= \alpha V^{t^{2i+1}} V^{t^{2i}} \\ \rho_u^{t^{2i+1}} &= \alpha V^{t^{2i+1}} V^{t^{2i+2}}.\end{aligned}$$

We now expand the expression

$$\rho_u^{-1} \rho_u^t \left(\rho_u^{t^2}\right)^{-1} \rho_u^{t^3} \cdots \left(\rho_u^{t^{2n-2}}\right)^{-1} \rho_u^{t^{2n-1}}, \quad (1)$$

noting that all the α 's cancel and that $\rho_u^{t^{2n-1}} = \alpha V^{t^{2n-1}} V^{t^{2n}} = \alpha V^{t^{2n-1}} V$ by Corollary 3. Expression (1) then becomes

$$V^{-1} \left(V^t\right)^{-1} \cdot V^t V^{t^2} \cdot \left(V^{t^2}\right)^{-1} \left(V^{t^3}\right)^{-1} \cdots \left(V^{t^{2n-2}}\right)^{-1} \left(V^{t^{2n-1}}\right)^{-1} \cdot V^{t^{2n-1}} V = \text{Id}.$$

Now by Lemma 5, the even transpose powers appearing in Expression (1) commute past the odd ones, and we can use Lemma 4 to translate these back into multiplication in R .

$$\begin{aligned}
\left[\rho_u^{-1} \left(\rho_u^{t^2} \right)^{-1} \cdots \left(\rho_u^{t^{2n-2}} \right)^{-1} \right] \left[\rho_u^t \rho_u^{t^3} \cdots \rho_u^{t^{2n-1}} \right] &= \text{Id} \\
\rho_u^t \rho_u^{t^3} \cdots \rho_u^{t^{2n-1}} &= \rho_u^{t^{2n-2}} \cdots \rho_u^{t^2} \rho_u \\
\rho_u^t \rho_u^{t^3} \cdots \rho_u^{t^{2n-1}} r &= \rho_u^{t^{2n-2}} \cdots \rho_u^{t^2} \rho_u r \quad (\forall r \in R) \\
u \sigma(u) \cdots \sigma^{n-1}(u) r &= r u \sigma(u) \cdots \sigma^{n-1}(u) \quad (\forall r \in R) \\
N_\sigma(u) r &= r N_\sigma(u) \quad (\forall r \in R)
\end{aligned}$$

so $N_\sigma(u) \in Z(R)$, as desired. \square

To illustrate the conjecture of the converse, we offer an algebra with two forms that are not homothetic for which the norm of the unit separating them is not central.

Example 17 Consider the extended Nakayama-Nesbitt algebra

$$R := \mathbb{C}\langle x, y \rangle / (x^2 = y^2 = 0, xyx = yxy),$$

a six-dimensional algebra with \mathbb{C} -basis $\{1, x, y, xy, yx, xyx\}$. Then R possesses a pair of forms that are not mutually homothetic, separated by a unit whose norm is not central.

PROOF. As with the original Nakayama-Nesbitt example, the functional $\lambda(a + bx + cy + dxy + eyx + fxyx) := f$ and the corresponding form $B(r, s) = \lambda(rs)$ show that R is Frobenius. The Nakayama automorphism is $\sigma : x \mapsto y, y \mapsto x$, because, for example $B(x, yx) = 1 = B(yx, y)$, so $y = \sigma(x)$. Then, of course, $\sigma^2 = \text{Id}$.

Fix $\epsilon \in \mathbb{C}$ and let $u := 1 + \epsilon x \in U(R)$. Then we have a form B_ϵ defined by $B_\epsilon(r, s) := B(r, su)$. Fixing the ordered basis $\{1, x, y, xy, yx, xyx\}$, we abuse the notation slightly and think of elements of R as column vectors and B_ϵ as a square matrix. Then $B_\epsilon(r, s) = r^T B_\epsilon s$, where T denotes ordinary matrix transposition. The matrix for B_ϵ is

$$B_\epsilon = \begin{pmatrix} 0 & 0 & 0 & \epsilon & 0 & 1 \\ 0 & 0 & \epsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \epsilon & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For example, there is an ϵ in the (2, 3) position because

$$B_\epsilon(x, y) = B(x, y(1 + \epsilon x)) = \lambda(xy(1 + \epsilon x)) = \lambda(xy + \epsilon xyx) = \epsilon.$$

We can use the matrix to calculate the new Nakayama automorphism in matrix form, which we will denote by Σ_ϵ :

$$s^T B_\epsilon \Sigma_\epsilon r = r^T B_\epsilon s = s^T B_\epsilon^T r \quad (\forall r, s \in R),$$

$$\text{so } \Sigma_\epsilon = B_\epsilon^{-1} B_\epsilon^T.$$

We claim, and we will prove below, that the forms for $\epsilon = 0$ and $\epsilon \neq 0$ are not homothetic. Conjecture 16 then predicts that $N_\sigma(u) \notin Z(R)$ for $\epsilon \neq 0$, and indeed,

$$N_\sigma(u) = u\sigma(u) = (1 + \epsilon x)(1 + \epsilon y) = 1 + \epsilon x + \epsilon y + \epsilon^2 xy \notin Z(R)$$

because, for example, it does not commute with x .

To prove the claim, we note that if a form B' is homothetic to B , then the matrix for its Nakayama automorphism must be similar to the matrix for the automorphism for B . This is because if $B'(r, s) = \alpha B(Vr, Vs) \forall r, s \in R$, then $B' = \alpha V^T B V$, so

$$(B')^{-1} (B')^T = V^{-1} B^{-1} (V^T)^{-1} V^T B^T V = V^{-1} B^{-1} B^T V.$$

However, it may be checked (by comparing Jordan Canonical Forms) that the matrix $B_\epsilon^{-1} B_\epsilon^T$ when $\epsilon \neq 0$ is *not* similar to the matrix $B_\epsilon^{-1} B_\epsilon^T$ when $\epsilon = 0$; therefore the forms are not homothetic. This confirms Conjecture 16 for this particular pair of forms. \square

We note that although the unit above does not have a central norm, the algebra does contain nontrivial units with central norms. If we take $u = 1 + \epsilon x - \epsilon y$ for some $\epsilon \in \check{C}$, then

$$N_\sigma(u) = (1 + \epsilon x - \epsilon y)(1 - \epsilon x + \epsilon y) = 1 + \epsilon^2 xy + \epsilon^2 yx \in Z(R).$$

If Conjecture 16 is true, then the form B' defined by u should be homothetic to B . And indeed, this turns out to be true, although we suppress the somewhat laborious calculations necessary to confirm this.

6 Determinants of forms

Although the results above show that the homothety class of a form on an algebra may not be well-defined, we will show now that for even-dimensional

local Frobenius algebras, the determinant of a form is an invariant for the algebra.

We first note that if B is a form on the Frobenius k -algebra R , then $\det B$ is a well-defined element of the square class group \dot{k}/\dot{k}^2 . To see this, fix an ordered k -basis for R and, by abuse of notation, write B for the matrix of the form with respect to this basis. If we use the map $V \in \text{Aut}_k R$ to change bases, then the form $B(Vr, Vs)$ will have matrix $V^T B V$, which has the same determinant as B in \dot{k}/\dot{k}^2 .

Now suppose B' is another form on R separated from B by the unit $u \in U(R)$, so $B'(r, s) = B(r, su) \forall r, s \in R$. Writing ρ_u for right multiplication by u , the new form has matrix $B' = B\rho_u$, so $\det B' = \det B \det \rho_u$.

If $u = \alpha \in \dot{k}$, then $\rho_u = \alpha \text{Id}$, which has determinant α^m , where $m = \dim_k R$. If m is odd, then $\overline{\alpha^m} = \bar{\alpha} \in \dot{k}/\dot{k}^2$, so $\det B'$ could be anything and it is pointless to hope for any significance to the determinant. In the even-dimensional case, though, it turns out to be an invariant for the algebra:

Theorem 18 *Let k be a field of arbitrary characteristic and let (R, \mathfrak{m}) be an even-dimensional local Frobenius k -algebra with $R/\mathfrak{m} \simeq k$ and form B . Then $\det B$, valued in \dot{k}/\dot{k}^2 , is independent of the choice of the form. Given another even-dimensional local Frobenius algebra (R', \mathfrak{m}') with $R'/\mathfrak{m}' \simeq k$ and form B' , we have $\det B = \det B'$ in \dot{k}/\dot{k}^2 if R and R' are isomorphic as k -algebras.*

PROOF. By the discussion above it suffices to show that $\det \rho_u$ is a square in \dot{k} . We will do this essentially by showing that the matrix for ρ_u with respect to a suitable basis is upper triangular. Suppose $\mathfrak{m}^n \neq 0 = \mathfrak{m}^{n+1}$. Construct an ordered basis $\{e_i\}$ for R as follows: start with a basis for \mathfrak{m}^n ; then complete it to a basis for \mathfrak{m}^{n-1} , and so on.

Let $u = \alpha(1 + m)$, where $\alpha \in \dot{k}, m \in \mathfrak{m}$. Then we may assume that $\alpha = 1$, since multiplying ρ_u by a scalar matrix will only change its determinant by a square.

In the matrix of ρ_u with respect to the basis $\{e_i\}$, the i -th column is given by the coordinates of $\rho_u e_i$ with respect to the basis. Now

$$\rho_u e_i = e_i u = e_i(1 + m) = e_i + e_i m.$$

Since $e_i m$ lies in a strictly higher power of \mathfrak{m} than e_i , it is in the span of basis vectors e_j with $j < i$. Hence the i -th column of ρ_u has a 1 in the i -th row, 0's below the i -th row, and other undetermined entries above. So ρ_u is upper triangular with 1's on the diagonal. So $\det \rho_u = 1$, as desired. \square

Example 19 Let $a, b, c \in \dot{k}$ with $b^2 \neq ac$, and let R be the four-dimensional local commutative k -algebra

$$R := k[x, y]/(ax^2 = bxy = cy^2, (x, y)^3 = 0) = k \oplus kx \oplus ky \oplus kx^2.$$

Then the determinant of any form on R , up to square, is $\delta(R) = ac(ac - b^2)$.

Thus, if two such algebras have different values for δ (taken in the square class group \dot{k}/\dot{k}^2) we know immediately that they are not isomorphic.

PROOF. We have a form $B(r, s) = \lambda(rs)$, where $\lambda(a + bx + cy + dx^2) := d$. The matrix for B with respect to the ordered basis $\{1, x, y, x^2\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{a}{b} & 0 \\ 0 & \frac{a}{b} & \frac{a}{c} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The determinant is $a^2/b^2 - a/c$, which in the square class group is equal to $ac(ac - b^2)$. \square

In this example, the converse is true too: the value of δ in \dot{k}/\dot{k}^2 actually determines the algebra up to isomorphism. To see this, consider the algebra

$$R' := k[u, v]/(u^2 = -\delta v^2, uv = 0, (u, v)^3 = 0) = k \oplus ku \oplus kv \oplus ku^2,$$

which is isomorphic to R under the map $u \mapsto \delta y, v \mapsto abx - acy$. It is then clear that the isomorphism type of R' is unchanged if the value of δ is multiplied by a nonzero square.

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