CECS 528, Learning Outcome Assessment 6, Spring 2024, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems

LO2. Solve the following problems.

(a) Use the Master Theorem to determine the growth of $T(n)$ if it satisfies the recurrence $T(n) = 4T(n/2) + n^{\log_4 16}$. Defend your answer.

Solution. $n^{\log_b a} = n^{\log_2 4} = n^2$

$f(n) = n^{\log_4 16} = n^2$. $\therefore$ By Case 2 of M.T., $T(n) = \Theta(n^2 \log n)$

(b) Use the substitution method to prove that if $T(n)$ satisfies

$$T(n) = 4T(n/2) + 3n^2 + 2n$$

then $T(n) = \Omega(n^2)$.

Solution. Inductive Assumption: $T(k) \geq ck^2$ for all $k<n$ and some const. $c>0$

Show $T(n) \geq cn^2$.

$T(n) = 4T(n/2) + 3n^2 + 2n \geq 4\left(\frac{cn}{2}\right)^2 + 3n^2 + 2n =$

$cn^2 + 3n^2 + 2n \geq cn^2 \Leftrightarrow 3n^2 + 2n \geq 0$

which is true since we are assuming $n \geq 0$.

LO3. Solve the following problems.

(a) Recall that the `find_statistic` algorithm makes use of Quicksort's partitioning algorithm and uses a pivot that is guaranteed to have at least

$$3(\lfloor \frac{1}{2}\lceil \frac{n}{5}\rceil \rfloor - 2) \geq 3(\frac{1}{2}\cdot\frac{n}{5} - 3) = \frac{3n}{10} - 9 \geq n/4$$

3 is replaced with 7 since, if pivot M is $\geq$ a group median then it is also $\geq$ 6 other members of that group: $1+6=7$.

members of a on both its left and right sides, assuming $n \geq 200$. Rewrite all three inequalities/equalities with updated constants, assuming that the algorithm now uses groups of 13 instead of groups of 5. Give the rationale for how you decided to replace the 3 on the left side of the very first inequality.

Solution. $7(\lfloor \frac{1}{2}\lceil \frac{n}{13}\rceil \rfloor - 2) \geq 7(\frac{1}{2}\cdot\frac{n}{13} - 3) =$

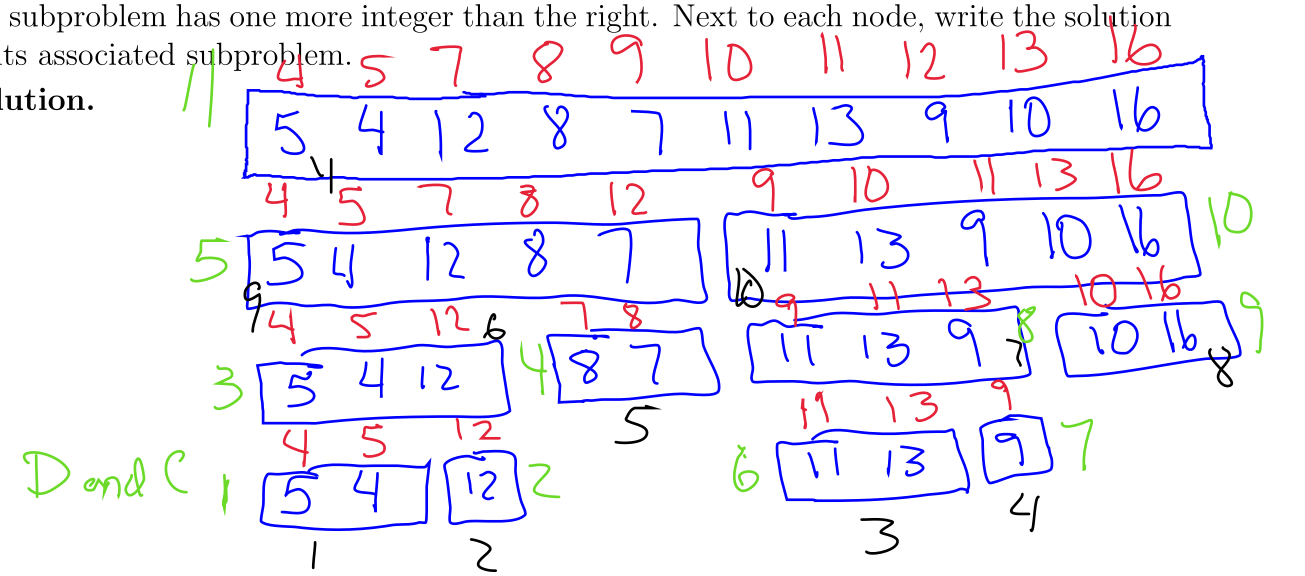
$\frac{7n}{26} - 21 \geq \frac{n}{4} \Leftrightarrow \frac{7n}{26} - \frac{n}{4} \geq 21 \Leftrightarrow (\frac{14n}{52} - \frac{13n}{52}) \geq 21 \Leftrightarrow n \geq (21)(52) = 1092$

(b) Draw the recursion tree that results when applying Mergesort to the array

$$a = 5, 4, 12, 8, 7, 11, 13, 9, 10, 16,$$

Label each node with the sub-problem to be solved at that point of the recursion. Assume arrays of size 1 and 2 are base cases. Assume that odd-sized arrays are split so that the left subproblem has one more integer than the right. Next to each node, write the solution to its associated subproblem.

Solution.



LO4. Answer the following.

(a) The FFT algorithm owes its existence to what two properties that are possessed by the nth roots of unity when n is even?

Solution. i) The n^{th} roots of unity come in additive-inverse pairs: $\pm\omega_n^j$ and so squaring them yields $\frac{n}{2}$ squares which ii) represent the $n/2$-roots of unity.

(b) Compute $\mathrm{DFT}_4^{-1}(4, -3, 2, -1)$ using the IFFT algorithm. Show the solution to each of the seven subproblem instances and, for each one, clearly represent it using DFT^{-1} notation and apply the formula for computing it. Show all work.

Solution.

$$\mathrm{DFT}_4^{-1}(4,-3,2,-1) = \frac{1}{2}\left[(3,1,3,1) + (1,-i,-1,i)\odot(-2,-1,-2,-1)\right] = \boxed{\left(\frac{1}{2}, \frac{1+i}{2}, \frac{5}{2}, \frac{1-i}{2}\right)}$$

$$\mathrm{DFT}_2^{-1}(4,2) = \frac{1}{2}\left[(4,4) + (1,-1)\odot(2,2)\right] = (3,1)$$

$$\mathrm{DFT}_2^{-1}(-3,-1) = \frac{1}{2}\left[(-3,-3) + (1,-1)\odot(-1,-1)\right] = (-2,-1)$$

$\mathrm{DFT}_1^{-1}(4) = 4$ $\quad\mathrm{DFT}_1^{-1}(2) = 2$ $\quad\mathrm{DFT}_1^{-1}(-3) = -3$ $\quad\mathrm{DFT}_1^{-1}(-1) = -1$

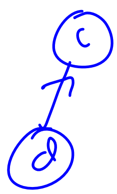
LO5. For the weighted graph with edges

$$(a, e, 4), (b, c, 6), (b, e, 3), (c, d, 1), (d, e, 2), (d, f, 5),$$

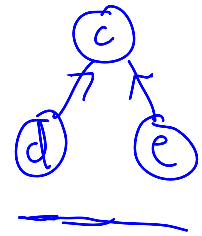
Show how the disjoint-set data structure forest changes when processing each edge in Kruskal's sorted list of edges. When unioning two trees, use the convention that the root of the union is the root which has the *lower* alphabetical order. For example, if two trees, one with root a, the other with root b, are to be unioned, then the unioned tree should have root a. Hint: remember that the algorithm terminates once a single tree remains.

Solution.

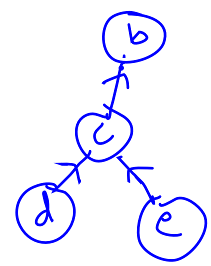
1. $(c, d, 1)$



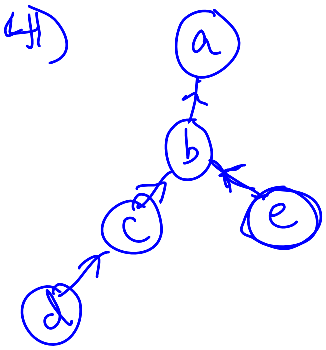
2. $(d, e, 2)$



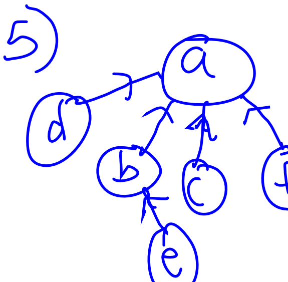
3. $(b, e, 3)$



4. $(a, e, 4)$



5. $(d, f, 5)$



LO6. Answer the following with regards to a correctness-proof outline for the Task Selection algorithm (TSA).

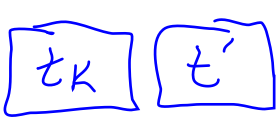
(a) Let $T = t_1, \ldots, t_m$ be the set of non-overlapping tasks selected by TSA and sorted by finish time, i.e. $f(t_i) < f(t_{i+1})$ for all $i = 1, \ldots, m-1$. Let T_{opt} be an optimal set of tasks and assume that, for some $k \geq 1$, $t_1, \ldots, t_{k-1} \in T_{\text{opt}}$, but $t_k \notin T_{\text{opt}}$. Explain why there must be at least one task $t' \in T_{\text{opt}}$ that overlaps with t_k. Hint: "Because if there was no such task ...".

Solution. $T_{opt} = t_1, \ldots, t_{k-1}, \underset{\underset{t'}{\parallel}}{t'_1}, t'_2, \ldots, t'_n$

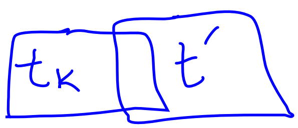
Otherwise T_{opt} would not be optimal: $|T_{opt}| = k-1 < k = |T|$

(b) Explain why there is *at most* one task $t' \in T_{\text{opt}}$ that overlaps with t_k. Hint: assume there are two overlapping tasks, t' and t'', and explain why this creates a contradiction.

Solution. $t_1, \ldots, t_{k-1}, t', t'_2, \ldots, t'_n$



incorrect since $T_{opt} + \{t_k\}$ would be optimal

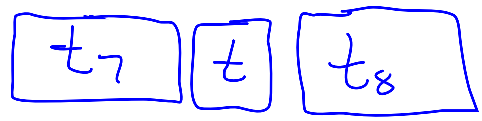


accurate picture

contradict t_k finishing earliest after t_{k-1}.

(c) Thus, we can define a new optimal set of tasks $\hat{T}_{\text{opt}} = T_{\text{opt}} - \{t'\} + \{t_k\}$ that contains $t_1, \ldots, t_k$. Continuing in this manner, we may obtain an optimal set of tasks T_{opt} for which $T \subseteq T_{\text{opt}}$. Moreover, we also have $T_{\text{opt}} \subseteq T$, since there is no way of add another task to T that does not overlap with one of T's tasks. For example, explain why it would not be possible to place a task t in between tasks t_7 and t_8. Therefore, we have established that $T = T_{\text{opt}}$ and TSA is correct.

Solution. Suppose $t \in T_{opt} - T$



Not possible since Algorithm would have chosen t instead of t_8.