

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems and Solutions

LO1. Complete the following problems.

(a) Evaluate $(-24)^{10} \pmod{7}$. Show steps.

$$(-24)^{10} \equiv (4)^{10} \equiv (2^2)^{10} \equiv 2^{20} \pmod{7}$$

$$2^3 \equiv 8 \equiv 1 \pmod{7}$$

$$2^{20} \equiv 2^2 \cdot (2^3)^6 \equiv 4 \pmod{7}$$

(b) For the Strassen-Solovay primality test, is $a = 5$ a witness or accomplice when $n = 9$? Show work and explain.

$$a^{\frac{9-1}{2}} \equiv \left(\frac{5}{9}\right) \pmod{9} ?$$

$$5^2 \equiv 25 \equiv 7 \pmod{9}$$

$$7^2 \equiv 4 \pmod{9}$$

$$5^4 \equiv \left(\frac{5}{9}\right) \pmod{9}$$

$$\left(\frac{5}{9}\right) = \left(\frac{9}{5}\right) = \left(\frac{4}{5}\right) = \left(\frac{2}{5}\right)^2 = 1$$

$$4 \not\equiv 1 \pmod{9} ?$$

∴ $a = 5$ is a witness

LO2. Complete the following problems.

- (a) Use the Master Theorem to determine the growth of $T(n)$ if it satisfies the recurrence $T(n) = 10T(n/3) + n^3$.

$$\underline{n^{\log_3 10} = o(n^3)}$$

By Case 3 of M.T. $T(n) = \Theta(n^3)$ for $\varepsilon = 3 - \log_3 10$

- (b) Use the substitution method to prove that, if $T(n)$ satisfies

$$T(n) = 4T(n/2) + 3n,$$

Then $T(n) = O(n^2)$. Hint: remember to state the inductive assumption.

Inductive assumption: $T(k) \leq Ck^2 + Dk$
for all $k < n$.

Show: $T(n) \leq Cn^2 + Dn$.

$$T(n) = 4T\left(\frac{n}{2}\right) + 3n \leq 4\left(C\left(\frac{n}{2}\right)^2 + D\left(\frac{n}{2}\right)\right)$$

$$+ 3n = Cn^2 + 2Dn + 3n \leq Cn^2 + Dn$$

$$\Leftrightarrow Dn \leq -3n \Leftrightarrow D \leq -3$$

~~1~~