

# Approximation Algorithms

Last Updated: December 5th, 2023

## 1 Introduction

In the previous lecture we saw how there exist many optimization problems that cannot be solved in polynomial time. For if the optimization problem is solvable in polynomial time, then so is its corresponding decision problem, and vice versa. We now return to the study of optimization problems, and ask if there are polynomial-time algorithms that can at least provide an approximation of the optimal solution.

## 2 Approximation Ratio

Let  $L$  be a minimization problem and suppose  $\mathcal{A}$  is an approximation algorithm for  $L$ . Let  $A(x)$  denote the (numerical) value returned by the algorithm on instance  $x$  of  $L$ , while  $\text{OPT}(x)$  denotes the optimal value for instance  $x$ . Then the **approximation ratio** of  $\mathcal{A}$  is defined as

$$\max_{x \in L} \frac{A(x)}{\text{OPT}(x)}.$$

Similarly, if  $L$  is a maximization problem, then the **approximation ratio** of  $\mathcal{A}$  is defined as

$$\max_{x \in L} \frac{\text{OPT}(x)}{A(x)}.$$

In the next sections we provide algorithms that give approximation-ratio upper bounds on four different intractable problems: Minimum Vertex Cover, Clustering, Load Balancing, and TSP.

### 3 An Approximation Algorithm for Minimum Vertex Cover

Consider the following approximation algorithm for the Minimum Vertex Cover optimization problem. Each step the algorithm randomly selects an edge from the current graph, and adds the edge vertices to the cover. Indeed, given current graph  $G_i = (V_i, E_i)$ , at Step  $i$  an edge  $e$  is selected from  $G_i$ . Then  $G_{i+1}$  is formed by removing from  $G_i$  both vertices  $u$  and  $v$  that are incident with  $e$ , and all edges that are incident with either  $u$  or  $v$ . Both  $u$  and  $v$  are then added to the cover. The algorithm terminates at Step  $i$  iff  $E_i = \emptyset$ . Moreover, the algorithm begins with  $G_1 = G$ , where  $G$  is the input graph.

**Example 3.1.** Demonstrate the above algorithm on the graph whose edges are

$\checkmark$   $\times$   $\checkmark$   $\times$   $\times$   $\times$   $\times$   $\checkmark$   $\times$   $\times$   $\times$   $\times$   $\times$   $\times$   $\times$   $\times$   $\times$   
(6, 8), (4, 8), (2, 5), (4, 5), (7, 8), (5, 7), (8, 2), (1, 4), (4, 3), (6, 3), (6, 7), (4, 7), (5, 6), (5, 3), (1, 7), (1, 3)

Compare the size of the minimum cover with that returned by the algorithm.

Cover  $C = \{6, 8, 2, 5, 1, 4\}$

$|C| = 6$

$\frac{6}{5} = 1.2$

Exercise: show that the graph has an optimal cover size equal to 5

**Theorem 3.2.** The above algorithm has an approximation ratio equal to 2.

**Proof.** At each step of the algorithm at least one of the vertices incident with selected edge  $e$  must be in the optimal cover. This is true since *every* edge of  $G$  must be incident with at least one vertex of the optimal cover. Therefore, since two vertices are added at each step, the worst case is that, at each step, exactly one of the vertices is not part of the optimal cover, which yields a cover size that is twice the size of the optimal one. Note that the worst case occurs when  $G$  has exactly one edge.  $\square$

In the Greedy-Algorithm lecture one of the exercises considered the following greedy heuristic for finding a minimum vertex cover. At each step, select the vertex  $v$  from  $G_i$  having highest degree and add  $v$  to the cover. Then  $G_{i+1}$  is obtained from  $G_i$  by removing  $v$  and any edge incident with  $v$ .

**Theorem 3.3.** There is a graph of order  $\Theta(n \log n)$  for which the above greedy heuristic produces a cover that is a factor  $\Theta(\log n)$  greater in size than the minimum cover.

**Proof.** Given positive integer  $n$ , consider the bipartite graph with vertex set  $W = \{w_1, \dots, w_n\}$  left side and vertex sets  $U_1, \dots, U_n$  on the right, where  $U_i = \{u_{i1}, \dots, u_{i\lfloor n/i \rfloor}\}$ . For the edges of the graph,  $w_k$  is adjacent to  $u_{ij}$  iff

$$\lfloor k/i \rfloor = j.$$

We have the following facts about this graph.

1. The left side has  $n$  vertices, while the right side has

$$n + \lfloor n/2 \rfloor + \lfloor n/3 \rfloor + \dots + \lfloor n/n \rfloor = \Theta(n \log n)$$

vertices.

2. The values of  $k$  that satisfy this equation

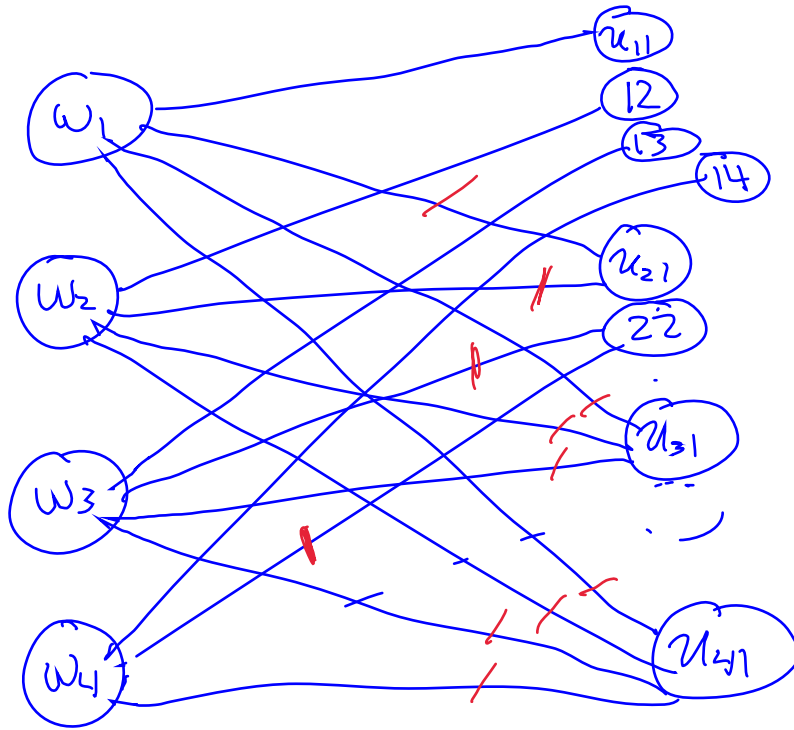
$$\lfloor k/i \rfloor = j.$$

are  $k = ij, ij - 1, \dots, ij - i + 1$ , implying that  $u_{ij}$  has degree  $i$ .

3. Each  $w_k$  is adjacent to at most one vertex in  $U_i$ , since the equation  $\lfloor k/i \rfloor = j$  implies a unique  $j$  value.
4. From Fact 3, for all  $1 \leq i \leq n$ , the subgraph induced by vertices  $W, U_1, \dots, U_i$  is such that each vertex in  $W$  has degree at most  $i$ .
5. From Fact 4, if the heuristic breaks ties by selecting vertices from the  $U$  sets, then the heuristic will select all vertices in  $U_n$ , followed by all vertices in  $U_{n-1}$ , etc., all the way down to selecting vertices from  $U_1$ . In other words, the heuristic selects  $\Theta(n \log n)$  vertices, yielding a  $\Theta(\log n)$  approximation ratio. □

$$(w_k, u_{ij}) \in E \text{ iff } \lfloor \frac{k}{i} \rfloor = j$$

Example 3.4. Verify Theorem 3.3 for  $n = 4$ .



$$k=4$$

$$i=3$$

$$\lfloor \frac{4}{3} \rfloor = 2$$

$$k=3$$

$$i=3$$

$$C = \{ u_{41}, u_{31},$$

$$u_{21}, u_{22},$$

$$u_{11}, u_{12}, u_{13}, u_{14} \}$$

# 4 An approximation algorithm for Clustering

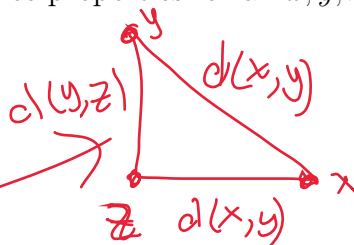
## 4.1 Preliminary definitions

**Definition 4.1.** A **metric space** is a pair  $(X, d)$ , where  $X$  is a set of “points” and  $d : X \times X \rightarrow R$  is a nonnegative function that satisfies the following three properties for all  $x, y, z \in X$ .

**Nonnegative**  $d(x, y) \geq 0$  and  $d(x, y) = 0$  iff  $x = y$

**Symmetric**  $d(x, y) = d(y, x)$

**Triangle Inequality**  $d(x, z) \leq d(x, y) + d(y, z)$



For example,  $(R^n, d)$  is a metric space, where  $R^n$  is the set of all  $n$ -dimensional real-valued vectors, and  $d(\vec{x}, \vec{y})$  denotes the Euclidean distance between  $\vec{x}$  and  $\vec{y}$ .

**Definition 4.2.** Given a metric space  $(X, d)$  and  $A \subseteq X$ , the **diameter** of  $A$ , denoted  $\text{diam } A$  is defined as

$$\text{diam } A = \sup\{d(x, y) \mid x, y \in A\},$$

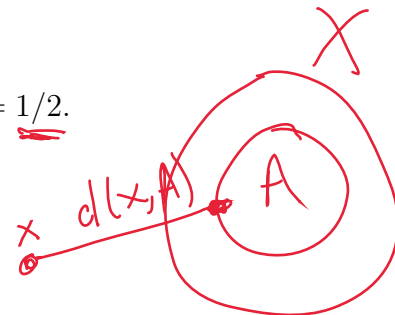
where  $\sup B$  denotes the least upper bound for a set of numbers  $B$ . Note that, in case  $A$  is finite, then

$$\text{diam } A = \max_{a_1, a_2 \in A} d(a_1, a_2).$$

For example,  $\text{diam } \{1, \dots, 100\} = 100 - 1 = 99$ , while  $\text{diam } \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\} = \frac{1}{2}$ .

Given a metric space  $(X, d)$ ,  $x \in X$ , and  $A \subseteq X$ , we define

$$d(x, A) = \inf\{d(x, y) \mid y \in A\},$$



where  $\inf B$  denotes the greatest lower bound for a set of numbers  $B$

For example,  $d(0, \{1, \dots, 100\}) = |0 - 1| = 1$ , while  $d(0, \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}) = 0$ .

**Definition 4.3.** For  $k \geq 1$ , A **k-partition** of a set  $S$  is a collection  $C_1, \dots, C_k \subseteq S$  of nonempty subsets of  $S$  that satisfies the two properties

1.  $C_i \cap C_j = \emptyset$  for all  $i \neq j$ ,
2.  $C_1 \cup \dots \cup C_k = S$ .

For example,  $\{1, \dots, 25\}$ ,  $\{26, \dots, 48\}$ , and  $\{49, \dots, 100\}$  is a 3-partition of  $\{1, \dots, 100\}$ .

**Definition 4.4.** An instance of the **Clustering** optimization problem is a set of points  $S = \{x_1, \dots, x_n\}$  from a metric space  $(X, d)$  and a nonnegative integer  $k \geq 1$ . The problem is to determine  $k$ -partition of  $C_1, \dots, C_k$  for which

$$\max_j \text{diam } C_j$$

is a minimum amongst all possible  $k$ -partitions of  $S$ .

Note that the decision version of **Clustering** is NP-complete.

## 4.2 The Algorithm

Input: i) set of points  $S = \{x_1, \dots, x_n\}$  from a metric space  $(X, d)$ , ii)  $k \geq 1$ .

Output:  $k$ -partition  $C_1, \dots, C_k$ .

Initialize  $P = \{x_1\}$ .

While  $|P| < k$ ,

$P += \{x\}$ , where

$$x = \arg \max_{y \in S} d(y, P).$$

// $x$  is the furthest away from the members of  $P$

Initialize  $\mathcal{C} = \emptyset$ .

For each  $x \in P$

$\mathcal{C} += C_x$ , where  $C_x = \{y \in S \mid d(y, x) < d(y, P - \{x\})\}$

//Cluster  $C_x$  contains all points in  $S$  that are closer to  $x$  than to any other point in  $P$ .

Return  $\mathcal{C}$



**Example 4.5.** Draw a random set  $S$  of points and apply the algorithm to  $S$  and  $k = 4$ .

**Theorem 4.6.** The `Clustering` algorithm has an approximation ratio that is bounded by 2.

**Proof.** We may assume that  $|S| > k$ , since otherwise the algorithm returns an optimal set of clusters, each with diameter equal to zero. Let  $P$  be the set of centers and consider the point  $x$  which satisfies

$$x = \arg \max_{y \in S} d(y, P).$$

In other words,  $x$  would be the next center chosen if  $k + 1$  clusters were sought. Let  $r = d(x, P)$  denote the distance from  $x$  to the closest center. Then every other point must be within  $r$  of a center since  $x$  was chosen to be the furthest from all the centers. Hence, for two points  $y$  and  $z$  that belong to a cluster with center  $c$ , we have by the triangle inequality

$$d(y, z) \leq d(y, c) + d(c, z) \leq r + r = 2r,$$

and so every cluster has a diameter at most  $2r$ .

Now that for each pair of points  $y, z \in P \cup \{x\}$ ,  $d(y, z) \geq r$ . This is because  $d(x, P) \geq r$  and, each point  $y$  added to  $P$  in the `while`-loop must have satisfied the predicate  $d(y, P) \geq r$  since this predicate is satisfied by  $x$ , but  $x$  was never chosen, meaning that, if  $y$  was chosen instead,  $y$  must also satisfy the predicate since the chosen point is the one which maximizes  $d(\cdot, P)$ . Finally, any  $k$ -partition of  $S$  must put at least two points from  $P \cup \{x\}$  into the same cluster which thus will have a diameter of at least  $r$ .  $\square$

## 5 An approximation algorithm for Load Balancing

An instance of the Load Balancing (LB) optimization problem consists of a positive integer  $p$ , representing the number of processors, and a sequence  $t_1, \dots, t_n$  of times, where  $t_i$  denotes the amount of time needed for a processor to process the  $i$ th task. Each task is assigned to one of the processors and the processors work independently and in parallel in an attempt to complete all the tasks. Then the objective is to minimize the maximum time it takes for any one processor to complete all its assigned tasks.

We now describe an approximation algorithm for LB. To begin, let

$$T_{\text{tot}} = \sum_{i=1}^n t_i$$

denote the sum of all the task processing times. Let  $\eta \in (0, 1)$  be a constant, and call a task/time  $t_i$  *large* iff  $t_i \geq \eta T_{\text{tot}}$ , and *small* otherwise. Note that there are at most  $1/\eta$  large tasks (why?).

The first step of the approximation algorithm is to determine an optimal load balance in the case where only the large tasks are processed. Now, since there are a constant number of different ways to schedule the large tasks, this step can be done in constant time. Note however that this constant has a maximum value of  $p^{\frac{1}{\eta}}$  which can become very large when  $\eta$  is small.

The final step is to assign each of the small tasks, where the next small task is assigned to the processor that currently has been assigned the smallest load. Thus, the final step makes use of a greedy heuristic.

**Example 5.1.** Demonstrate the approximation algorithm for LB using the following tasks with  $p = 3$  processors and  $\eta = 1/12$ .

<b>Task</b>	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
<b>Duration</b>	1	7	7	6	7	8	6	2	8	10	4	5	10	9	6

**Theorem 5.2.** The above described algorithm provides a  $(1 + \eta(p - 1))$ -approximation for Load Balancing.

**Proof.** Let  $M_{\text{opt}}$  denote the optimal finish time for the last processor that completes its tasks. Let  $M_l$  denote the optimal finish time when only large tasks are scheduled. Note that, when adding the small tasks in the final step, if the latest processor finish time is not increased, then necessarily  $M_{\text{opt}} = M_l$ . So assume that the final step induces a latest finish time  $M > M_l$ . Let  $T_i$  denote the finish time of processor  $i$  after the final step. Without loss of generality, assume  $T_1 \geq \dots \geq T_p$ . Then at least one small task must have been added to  $T_1$  (why?). Let  $t$  denote the processing time for the last small task. assigned to the first processor. Then, at the time this task was assigned, it must have been true that

$$M - t \leq T_j \leq M$$

for all  $j = 2, \dots, p$ . Moreover, since  $t < \eta T_{\text{tot}}$ , we have

$$T_{\text{tot}} = M + \sum_{i=2}^p T_i \geq pM - (p - 1)t \geq pM - \eta(p - 1)T_{\text{tot}}.$$

Moreover, solving for  $M$  yields the inequality

$$M \leq \frac{T_{\text{tot}}}{p}(1 + \eta(p - 1)).$$

But, we also know that

$$M_{\text{opt}} \geq \frac{T_{\text{tot}}}{p},$$

since the righthand side represents the best-case scenario in which tasks are divided equally (in terms of processing time) by each of the processors and all processors work the same amount of time, and with no idle time. Thus,

$$M \leq M_{\text{opt}}(1 + \eta(p - 1))$$

which proves the theorem. Note also that  $M \geq pM_{\text{opt}}$  (why?), and so the two inequalities can be combined to get

$$p \leq \frac{M}{M_{\text{opt}}} \leq (1 + \eta(p - 1)).$$

□

The LB approximation algorithm above is what is called a **polynomial-time approximation scheme (pas)**, since for every  $\epsilon > 0$  we can adjust the value of  $\eta$  to  $\eta = \epsilon(p - 1)$  so that  $M \leq (1 + \epsilon)M_{\text{opt}}$ . Moreover, a **fully polynomial-time approximation scheme (fpas)** means that the algorithm runs in time polynomial with respect to the size of the problem, and with respect to  $1/\epsilon$ . The above algorithm is not an fpas, since the first step of the algorithm requires time  $p^{1/\eta}$ , and so grows exponentially with respect to  $1/\eta$  which is proportional to  $1/\epsilon$ .

## 6 An approximation algorithm for Traveling Salesperson

Recall that an input to the Traveling Salesperson (TSP) optimization problem is a complete weighted graph  $G = (V, E, c)$ , and the problem is to find a Hamilton cycle whose edge weights sum to a minimum value. We first prove the following.

**Theorem 6.1.** If there is a polynomial-time  $\rho$ -approximation algorithm for TSP, then  $\mathbf{P} = \mathbf{NP}$ .

**Proof.** Suppose there is a polynomial-time  $\rho$ -approximation algorithm  $A$  for TSP. Let  $G = (V, E)$  be a simple graph. To decide if  $G$  has a Hamilton cycle, we define the weighted graph  $G' = (V, E \cup \overline{E}, c)$ , where

$$c(e) = \begin{cases} 1 & \text{if } e \in E \\ 1 + \rho n & \text{otherwise} \end{cases}$$

where  $n = |V|$ . Then  $G$  has a Hamilton cycle iff  $G'$  has a cycle with minimum weight sum equal to  $n$ . Otherwise, the minimum weight sum of any Hamilton cycle is at least  $1 + \rho n$ . Thus, a polynomial-time  $\rho$ -approximation algorithm  $A$  would be able to distinguish between the two cases, since, in case  $G'$  has a cycle with minimum weight sum equal to  $n$ , then

$$A(G') \leq \rho n < 1 + \rho n.$$

Thus  $G$  has a Hamilton cycle iff  $A(G')$  returns a value not exceeding  $\rho n$ , which implies  $\mathbf{HC} \in \mathbf{P}$ . But if an NP-complete problem is in  $\mathbf{P}$ , then so is every other NP problem. Therefore,  $\mathbf{P} = \mathbf{NP}$ .  $\square$

It turns out that we can do better if we assume that the vertices of  $G$  lie in a **metric space**.

The following proposition provides the key idea behind the 2-approximation algorithm described below. We leave its proof as an exercise.

**Proposition 6.2.** Let  $G$  be a complete weighted graph whose edge weights satisfy the triangle inequality. Then given a path  $P = i_1, \dots, i_n$ ,  $n \geq 2$ , we have

$$d(i_1, i_n) \leq \sum_{j=1}^{n-1} d(i_j, i_{j+1}).$$

The following algorithm may be used to obtain a 2-approximation for TSP, assuming the input graph  $G$ 's edge weights satisfy the triangle inequality. First note that, if  $T$  is a minimum spanning tree for  $G$ , then the minimum-cost tour is at least equal to the cost of  $T$ . This is because any tour forms a single branch (and perhaps non-optimal) spanning tree from the start vertex to the final vertex. Moreover, the tour incurs the additional cost of returning to the start vertex from the final vertex. Now consider a path  $P$  that traverses only edges of  $T$ , visits every vertex (perhaps more than once), and returns to the start vertex. This can be accomplished by treating the start vertex as the root of  $T$ , performing a depth-first traversal along  $T$ , and eventually returning to the root vertex. The key observation about this depth-first traversal is that each tree edge is traversed exactly twice. Thus, the cost of  $P$  is twice the cost of  $T$ . Thus,  $P$  provides a 2-approximation of the minimum cost, but unfortunately  $P$  is not a Hamilton cycle. However,  $P$  can be converted to a Hamilton cycle without increasing the tour cost! To convert  $P$  to a Hamilton cycle, suppose  $i_1, \dots, i_k$ ,  $k \geq 3$ , is a subpath of  $P$  for which

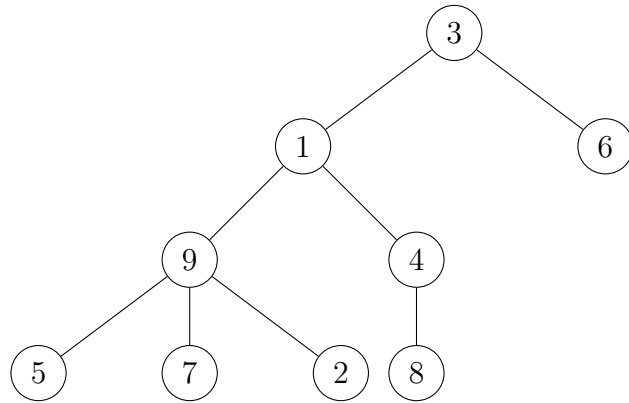
1. vertex  $i_1$  is making its first appearance in  $P$ ,
2. vertices  $i_2, \dots, i_{k-1}$  have already been visited in  $P$ , and
3. either  $i_k$  has yet to be visited in  $P$ , or  $i_k$  is the final vertex of  $P$  (and hence equals the start/root vertex).

Then we may replace the edges  $(i_1, i_2), \dots, (i_{k-1}, i_k)$  with the edge  $(i_1, i_k)$ , and, by Theorem 5, we have

$$d(i_1, i_k) \leq \sum_{j=1}^{k-1} d(i_j, i_{j+1}),$$

and so the revised path is no more costly than the original one. Finally, repeatedly replacing these subpaths eventually results in a Hamilton cycle which is a 2-approximation of the optimal Hamilton cycle.

**Example 6.3.** Suppose  $G$  is a weighted complete graph with vertex set  $V = \{1, \dots, 9\}$ , and for which a minimum spanning tree is shown below. Assuming the depth-first traversal visits children in the order from left to right, determine the Hamilton cycle produced by the above 2-approximation algorithm for TSP.





## An improved TSP approximation algorithm

Given a complete weighted graph  $G = (V, E, c)$ , Christophides's algorithm also uses a minimum-spanning tree  $T$  as the basis for an approximation algorithm for TSP. However, instead of performing a depth-first traversal on  $T$ , his algorithm instead adds edges to  $T$  so that every vertex has even degree. Moreover, since there are an even number of odd-degree vertices, this can be accomplished in polynomial time by finding a minimum-cost matching  $M$  over the odd-degree vertices of  $T$ . Let  $U = \{u_1, \dots, u_t\}$  denote the odd-degree vertices of  $T$ , and  $G_U$  denote the subgraph of  $G$  that is induced by  $U$ . Then  $M$  is a minimum-cost matching for  $G_U$ .

Now, once the  $M$ -edges have been added, one can perform an Euler tour, and then convert that tour to a Hamilton cycle by skipping already-visited vertices, as was done in the previous algorithm. Furthermore the cost of this cycle is no more than  $\text{cost}(T) + \text{cost}(M)$ . From the previous algorithm we already know that  $\text{cost}(T) \leq C_{\text{opt}}$ , where  $C_{\text{opt}}$  is the minimum cost of any TSP tour. What about  $\text{cost}(M)$ ? Consider a tour of  $G_U$  that uses the min-cost TSP tour of  $G$ , but skips all vertices that are not in  $U$ , then this tour cost does not exceed  $C_{\text{opt}}$ . Without loss of generality, assume that

$$u_1, u_2, \dots, u_{t-1}, u_t, u_1$$

is such a tour. Then this tour consists of two different matchings for  $G_U$ , namely  $M_1 = \{(u_1, u_2), \dots, (u_{t-1}, u_t)\}$ , and  $M_2 = \{(u_2, u_3), \dots, (u_t, u_1)\}$ . Thus, since

$$\text{cost}(M_1) + \text{cost}(M_2) \leq C_{\text{opt}},$$

and  $M$  is a minimum-cost matching for  $G_U$ , it follows that

$$\text{cost}(M) \leq \min(\text{cost}(M_1), \text{cost}(M_2)) \leq C_{\text{opt}}/2,$$

where the last inequality is due to the arithmetic fact that, if two numbers sum to a third number, then one of the two numbers in the sum cannot exceed one-half of the sum. Therefore

$$\text{cost}(T) + \text{cost}(M) \leq 3/2 C_{\text{opt}},$$

and Christophides's algorithm is a  $3/2$ -approximation algorithm for TSP.

**Example 6.4.** A delivery truck starts at intersection  $(0, 0)$ , and must deliver packages to intersections

$(4, 1)$ ,  $(2, 3)$ ,  $(5, 3)$ ,  $(0, 5)$ ,  $(2, 6)$ ,  $(4, 7)$ ,  $(6, 7)$ ,  $(3, 9)$ , and  $(0, 9)$ .

Provide two tours for the driver, using the 2-approximation and  $3/2$ -approximation algorithms, respectively. Compute the cost of each tour. Assume that vertex distances are measured via horizontal and vertical movement along the city grid, and that each grid segment takes unit time to traverse.

## Exercises.

1. Prove Theorem 5. Hint: use induction.

## Exercise Hints and Answers.

1. The proof is by induction on  $n \geq 2$ . The statement is trivial for  $n = 2$ , and is a re-statement of the triangle inequality for  $n = 3$ . Now assume it is true for any path that traverses  $n-1$  vertices, where  $n \geq 4$ , and consider path  $P = i_1, \dots, i_{n-1}, i_n$ . Then, by the inductive assumption, we have

$$d(i_1, i_{n-1}) \leq \sum_{j=1}^{n-2} d(i_j, i_{j+1}).$$

Thus, by the triangle inequality,

$$d(i_1, i_n) \leq d(i_1, i_{n-1}) + d(i_{n-1}, i_n) \leq \sum_{j=1}^{n-2} d(i_j, i_{j+1}) + d(i_{n-1}, i_n) = \sum_{j=1}^{n-1} d(i_j, i_{j+1}),$$

and the theorem is proved.