**CECS 419-519, Writing Assignment 7, Due 8:00 am, March 29th, 2024, Dr. Ebert**

# Directions

Make sure name is on all pages. Order pages (front and back) so that solutions are presented in their original numerical order. **Please no staples or folding of corners (your papers won't get lost). A paper clip is OK** Show all necessary work and substantiate all claims. Avoid plagiarism.

# Problems

1. Consider the following method for constructing a binary encoding of a probability distribution $p_1 \geq p_2 \geq \cdots \geq p_n$. Define $q_i$ by $q_1 = 0$ and, for $i \geq 2$,

$$q_i = p_1 + \cdots + p_{i-1}.$$

Let $m_i = \lceil \log(\frac{1}{p_i}) \rceil$. Finally, let codeword $c_i$ be the binary expansion of $q_i$ truncated at the first $m_i$ bits to the right of the binary point. For example, if $q_i = 37/64$ and $p_i = 1/20$, then $m_i = \lceil \log(20) \rceil = 5$ and $c_i$ is the first 5 bits of $q_i$, that is $c_i = 10010$, since $q_i = 100101$.

   (a) Apply the algorithm to the probability distribution $P = \{0.3, 0.3, 0.15, 0.10, 0.10, 0.05\}$. (10 pts)

   (b) Prove that the code obtained is a prefix code. (10 pts)

   (c) Prove that the code obtained has an average codeword length $A$ that satisfies

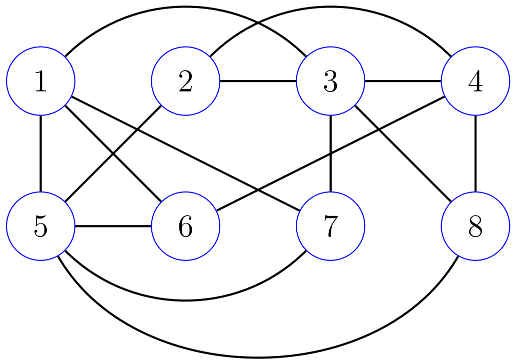
$$H_2(p_1, \ldots, p_n) \leq A \leq H_2(p_1, \ldots, p_n) + 1.$$

   (10 pts)

2. Recall the `Max Cut` decison problem from Writing Assignment 6. A special case of this problem is the `Half Max Cut` decision problem, where an instance is a graph $G$, and the problem is to decide if there is a way to color the vertices of $G$ using the colors red and blue and results in there being at least $\lfloor |V|/2 \rfloor$ edges $e = (u, v)$ for which $u$ and $v$ are assigned different colors.

   (a) Provide a complete description of a polynomial-time mapping reduction from `Max Cut` to `Half Max Cut`. Provide a convincing argument for why the reduction works. (20 pts)

   (b) Apply your reduction to the following instance $(G, k)$ of `Max Cut` where $G$ is the graph shown below and $k = 10$. (5 pts)




3. Given 3SAT instance

$$\mathcal{C} = \{(x_1, x_2, \overline{x}_5), (x_1, \overline{x}_3, \overline{x}_4), (x_1, x_4, x_5), (\overline{x}_1, x_2, x_4),$$

$$(\overline{x}_1, \overline{x}_2, \overline{x}_5), (\overline{x}_1, \overline{x}_2, x_5), (x_2, x_3, x_5), (x_2, \overline{x}_3, x_5),$$

$$(x_2, \overline{x}_4, \overline{x}_5), (\overline{x}_2, x_3, x_4), (\overline{x}_2, x_3, \overline{x}_5), (\overline{x}_2, \overline{x}_4, x_5),$$

$$(\overline{x}_2, x_3, \overline{x}_4), (x_3, \overline{x}_4, x_5), (\overline{x}_3, x_4, x_5), (\overline{x}_3, \overline{x}_4, \overline{x}_5)\},$$

answer the following questions.

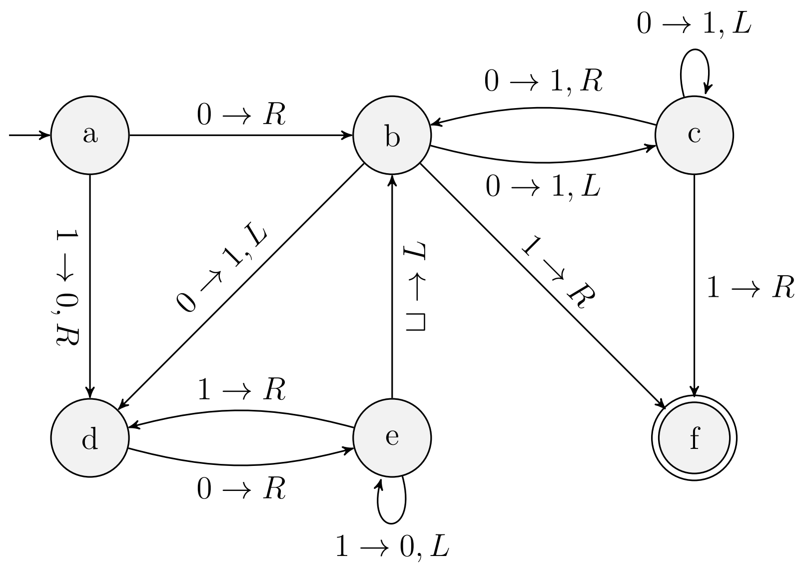
   (a) For the mapping reduction $f$ from `3SAT` to `Subset Sum` presented in lecture, if $f(\mathcal{C}) = (S, t)$, then what is $|S|$ and what is the largest number in $S$? Explain and show work. (10 pts)

   (b) Is there a subset $A$ of $S$ that sums to $t$? If yes, provide the members of $A$ (write each number using its identifier that is provided in the reduction). In any case, defend your answer. (10 pts)

   (c) For the mapping reduction $f$ from `3SAT` to `DHP` provided in lecture, describe the orientation of the edges $(\mathrm{lc}_8, c_8)$ and $(\mathrm{rc}_8, c_8)$, that connect Diamond 3 to clause vertex $c_8$. Explain. (5 pts)

4. Provide the state diagram for a Turing machine that, on input $x \in \{0, 1\}^+$, replaces $x$ with $x$ 1's. Here, we are thinking of $x$ as a binary number. For example, 1001 would be replaced with 111111111, while 00 is replaced with $\lambda$ (i.e. a tape with all blanks). Simulate your solution using the online Turing machine simulator and provide an in-class demo that includes an explanation of your algorithm. (20 pts)

5. For the Turing machine $M$ with state diagram shown below, draw the computation tree $T(M, 001)$. Is this an accepting computation? Explain. (15 pts)



a
b
c
d
e
f
$0 \to R$
$0 \to 1, R$
$0 \to 1, L$
$0 \to 1, L$
$1 \to 0, R$
$0 \to 1, L$
$\sqcup \to L$
$1 \to R$
$1 \to R$
$1 \to R$
$0 \to R$
$1 \to 0, L$