CECS 419-519, Writing Assignment 5, Due 8:00 am, March 1st, 2024, Dr. Ebert

Directions

Make sure name is on all pages. Order pages (front and back) so that solutions are presented in their original numerical order. **Please no staples or folding of corners (your papers won't get lost). A paper clip is OK** Show all necessary work and substantiate all claims. Avoid plagiarism.

Problems

1. Let $\mathcal{E}$ be a σ-algebra of sets. Show that if $A, B \in \mathcal{E}$, then $A - B \in \mathcal{E}$. (10 pts)

2. For the sample space $\{0,1\}^\infty$, recall the event space consisting of sets of the form $A \subseteq \{0,1\}^*$, where A is a finite minimal prefix code (see Example 1.5 of the Autoreducibility lecture). If $A = \{00, 010, 1011, 1101, 111\}$ and $B = \{000, 0110, 100, 1010, 11\}$, then provide minimal prefix codes C, D, and F, for which

   (a) $C\{0,1\}^\infty = \overline{A}\{0,1\}^\infty$,

   (b) $D\{0,1\}^\infty = A\{0,1\}^\infty \cup B\{0,1\}^\infty$, and

   (c) $E\{0,1\}^\infty = A\{0,1\}^\infty \cap B\{0,1\}^\infty$.

   (15 pts)

3. Prove that if a finite minimal prefix code A satisfies

$$\mu(A) = \sum_{w \in A} 2^{-|w|} = 1,$$

   then $A = \{\lambda\}$, where λ is the empty word. Hint: use mathematical induction on the length of the longest word(s) in A. (20 pts)

4. Let P be a probability measure over σ-algebra $\mathcal{E}$, and suppose $E_1 \subseteq E_2 \subseteq \cdots$ is a nested sequence of sets, where $E_i \in \mathcal{E}$, for $i = 1, 2, \ldots$. Provide a summation formula for computing $\mu(\bigcup_{i=1}^{\infty} E_i)$. Hint: you may find Problem 1 helpful. (15 pts)

5. Consider the set $\mathcal{B}$ of all infinite binary sequences s for which every length-n (n divisible by 4) prefix of s consists of exactly $3n/4$ 1's and $n/4$ 0's. Suppose we wish to make a null cover for $\mathcal{B}$ of the form

$$\mathcal{B} = \bigcap_{k=1}^{\infty} W_k\{0,1\}^\infty,$$

   where W_k consists of all binary words of some length n_k so that

(a) n_k is divisible by 4,

(b) $\mathcal{B} \subseteq W_k\{0,1\}^\infty$, and

(c) $\mu(W_k) \leq \frac{1}{2^k}$.

Provide a formula for computing an n_k that satisfies the above constraints. Hint: use the following result from combinatorics. For any n and $0 < \alpha < 1$,

$$\binom{n}{\alpha n} = \frac{1+o(1)}{\sqrt{2\pi\alpha(1-\alpha)n}} \cdot 2^{n \cdot H(\alpha)},$$

where $H(\alpha) = H(\alpha, 1-\alpha)$ is the entropy function. Your formula may ignore the $o(1)$ term since it becomes vanishingly small for increasing k. Is this null cover constructive? Explain. (20 pts)

6. For the binary matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

Determine its rank (i.e. dimension of its column space) and nullity (dimension of its NULL space) and verify the Rank theorem for A. List each of the null vectors (there are 16 of them). Can you see why the 1-balls centered around these null vectors i) do not overlap with each other and ii) cover the entire space $\{0,1\}^7$? Note that all arithmetic is over the field $\{0,1\}$ with $\oplus$ as addition: $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$. (20 pts)