

**NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED.** Solve each of six problems on a separate sheet of paper. For example, if you decide to solve all six problems, then you should submit six pages, one for each problem. Make sure your name is on each page.

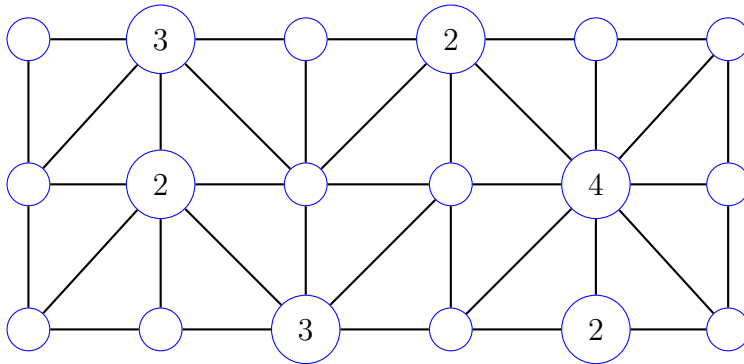
## Problems

1. Solve the following.
  - (a) Provide the definition of what it means to be a mapping reduction from decision problem  $A$  to decision problem  $B$ . Note: scoring 20 or more points is sufficient for passing LO1. (5 pts)
  - (b) Let  $S = \{6, 11, 13, 22, 23, 26, 27, 46\}$  be an instance of **Set Partition (SP)**. Provide  $f(S)$ , where  $f : \text{SP} \rightarrow \text{SS}$  is the mapping reduction from SP to **Subset Sum** provided in lecture. (10 pts)
  - (c) Using the problem instances in part b, verify that  $f$  maps a positive instance of SP to a positive instance of SS. (10 pts)

2. An instance of the **MineSweep** decision problem is a simple graph  $G = (V, E)$  and a map  $\text{val} : V \rightarrow \mathcal{N}$  for which some of  $G$ 's vertices are labeled with a positive integer. Namely, if  $\text{val}(v) = 0$ , then  $v$  is not labeled, but if  $\text{val}(v) > 0$ , then  $\text{val}(v)$  gives the value of the label. Note: correctly solving this problem counts for passing LO2.

The problem is to decide if there is a way to place mines on some of  $G$ 's unlabeled vertices so that, for each  $v \in V$  that is labeled with some integer  $k = \text{val}(v)$ , exactly  $k$  of  $v$ 's neighbors have been assigned a mine.

- (a) Show that the following labeled graph is a positive instance of **MineSweep**. Do so by placing an "x" in each node that should have a mine. (5 pts)



- (b) For a given instance  $(G = (V, E), \text{val})$  of **MineSweep**, describe a certificate in relation to  $(G = (V, E), \text{val})$ . (5 pts)

- (c) Provide a semi-formal verifier algorithm that takes as input i) an instance  $(G = (V, E), \text{val})$  of **MineSweep**, and ii) a certificate for  $(G = (V, E), \text{val})$  as defined in part a, and decides if the certificate is valid for  $(G = (V, E), \text{val})$ . (10 pts)
- (d) Describe the running time of your verifier from part c using size parameters  $m = |E|$  and  $n = |V|$  and explain why it is a polynomial with respect to the size parameters. (5 pts)
3. Recall the mapping reduction  $f(\mathcal{C}) = (S, t)$ , where  $f$  maps an instance of **3SAT** to an instance of the **Subset** decision problem. Given **3SAT** instance

$$\mathcal{C} = \{(x_1, \bar{x}_2, x_3), (x_2, x_3, \bar{x}_4), (x_1, x_2, x_4), (\bar{x}_1, \bar{x}_3, \bar{x}_4)\}$$

answer the following questions about  $f(\mathcal{C})$ . Hint: to answer these questions you are *not* required to draw the table, but you might find it helpful. Note: scoring 18 or more points on this problem counts for passing LO3.

- (a) What is the value of  $t$ ? (5 pts)
- (b) How many numbers (counting repeats) are in  $S$ ? What is the largest (in terms of numerical value) number in  $S$ ? (10 pts)
- (c) Determine a satisfying assignment for  $\mathcal{C}$  and use it to identify a subset  $A$  of  $S$  that sums to  $t$ . List all the members of  $A$ . Hint: there are multiple possible answers, but the subset you choose must correspond with your chosen satisfying assignment. (10 pts)
4. Answer the following questions. Correctly answering at least two of the three is sufficient for passing LO4.

- (a) Provide the definition of what it means for a decision problem to be NP-complete. (6 pts)
- (b) One of the exercises from the complexity lecture establishes a polynomial-time mapping reduction from **Independent Set** to **Vertex Cover**, i.e.  $\text{IS} \leq_m^p \text{VC}$ . Using this reduction as the final link in the chain, provide the chain of polynomial-time mapping reductions that establishes the NP-completeness of **VC**, where each of the remaining reductions was provided in either the Mapping Reducibility or Complexity lecture. (7 pts)
- (c) Each of the following decision problems described below takes as input a Boolean formula  $F$ . Classify each one as either being in P, NP, or co-NP. (4 pts each)
- $F$  has at least two distinct satisfying assignments  $\alpha_1$  and  $\alpha_2$ .
  - $F$  is a **fallacy** meaning that it evaluates to 0 on every assignment  $\alpha$ .
  - An instance of this problem has a second input  $k > 0$  and the problem is to decide if  $F$ 's Tseytin encoding (from **SAT** to **3SAT**) has at least  $k$  3-SAT clauses.

5. An instance  $\mathcal{C}$  of the  $\neq$ -**SAT** decision problem is the same as that of **3SAT**, except now we seek a satisfying assignment  $\alpha$  which is a  $\neq$ -**assignment**, meaning that, for each clause  $c \in \mathcal{C}$ , there must be a literal  $l_1 \in c$  for which  $\alpha(l_1) = 1$  and a literal  $l_2 \in c$  for which  $\alpha(l_2) = 0$ . Solve the following.

- (a) Find a  $\neq$ -assignment  $\alpha$  that satisfies each of the following clauses.

$$\mathcal{C} = \{(\bar{x}_2, \bar{x}_3, \bar{x}_4), (\bar{x}_1, x_2, x_3), (x_1, x_3, \bar{x}_4), (\bar{x}_1, x_2, \bar{x}_4), (x_2, \bar{x}_3, \bar{x}_4)\}.$$

(5 pts)

- (b) Given assignment  $\alpha$ ,  $\bar{\alpha}$  is called the **complement** of  $\alpha$ , where, for any literal  $l$  that is assigned by  $\alpha$ ,

$$\bar{\alpha}(l) = 1 - \alpha(l),$$

meaning that  $\bar{\alpha}$  assigns each literal the complement value of what  $\alpha$  assigns it. For the assignment  $\alpha$  you provided in part a, verify that  $\bar{\alpha}$  is also a  $\neq$ -assignment for the set of clauses in  $\mathcal{C}$ . In general, explain why the following statement is always true: “if  $\alpha$  is a  $\neq$ -assignment for a set of clauses, then so is  $\bar{\alpha}$ ”. (5 points)

- (c) Now consider the following mapping reduction from 3SAT to  $\neq$ -SAT. Each clause  $c \in \mathcal{C}$ , where  $c = (l_1, l_2, l_3)$ , is mapped to the two clauses

$$f(c) = (l_1, l_2, z_c), (\bar{z}_c, l_3, g),$$

where  $z_c$  is a new Boolean variable that is specific to  $c$ , and  $g$  is a global Boolean variable that appears in the mapping of each  $c$ . Now suppose  $\mathcal{C}$  is a negative instance of 3SAT. We want to show that the above mapping can never result in a positive instance of  $\neq$ -SAT. To this end, let  $\alpha$  be an assignment over the variables of  $\mathcal{C}$  and suppose  $\alpha$  does *not* satisfy  $c = (l_1, l_2, l_3)$ . How must  $z_c$  and  $g$  be assigned in order to make a  $\neq$ -assignment for  $f(c)$ ? Explain. (5 pts)

- (d) Let  $\beta$  be the extension of  $\alpha$  defined in the previous part. In other words,  $\beta$  assigns the variables in  $\mathcal{C}$  in the same way as  $\alpha$ , but it also assigns all the local  $z$  variables and the global  $g$  variable. Furthermore, suppose  $\beta$  is a  $\neq$ -assignment for  $f(\mathcal{C})$ . Then by part b, it must also be true that  $\bar{\beta}$  is also a  $\neq$ -assignment for  $f(\mathcal{C})$  and that it assigns the variables of  $\mathcal{C}$  in the same manner as  $\bar{\alpha}$ . Moreover, since  $\mathcal{C}$  is unsatisfiable, there must be a clause  $c' = (l'_1, l'_2, l'_3)$  that is unsatisfied by  $\bar{\alpha}$ . Explain why this implies that  $\bar{\beta}$  cannot form a  $\neq$ -assignment for the two clauses in  $f(c')$ , thus creating a contradiction. (10 pts)

6. Recall that the 3-Dimensional Matching (3DM) decision problem takes as input three sets  $A$ ,  $B$ , and  $C$ , each having size  $n$ , along with a set  $S$  of triples of the form  $(a, b, c)$  where  $a \in A$ ,  $b \in B$ , and  $c \in C$ . We assume that  $|S| = m \geq n$ . The problem is to decide if there exists a subset of  $n$  triples (called a **matching**) from  $S$  for which each member from  $A \cup B \cup C$  belongs to exactly one of the triples.

- (a) Describe a valid mapping reduction  $f$  from 3DM to MineSweep (See Problem 2 of this exam). Your description must be as general as possible so that it can be applied to any 3DM instance. Also, you must argue that  $(A, B, C, S)$  is positive iff  $f(A, B, C, S)$  is positive. Hint: make the instance  $f(A, B, C, S)$  of MineSweep a bipartite graph with labels on one side. Clearly define the vertices, any labels on the vertices, and a rule for when two vertices are adjacent. (15 pts)
- (b) Demonstrate the reduction from part a using the 3DM instance  $(A, B, C, S)$ , where  $A = \{a, b, c\}$ ,  $B = \{1, 2, 3\}$ ,  $C = \{x, y, z\}$ , and

$$S = \{(b, 2, y), (b, 1, z), (a, 3, z), (c, 2, y), (a, 2, y), (a, 3, y), (c, 3, x), (c, 1, z), (b, 1, x)\}.$$

(10 pts)