CECS 329, Exam 1 Spring 2024, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Solve each of six problems on a separate sheet of paper. For example, if you decide to solve all six problems, then you should submit six pages, one for each problem. Make sure your name is on each page.

Problems

1. Solve the following.

   (a) Provide the definition of what it means to be a mapping reduction from decision problem A to decision problem B. Note: scoring 20 or more points is sufficient for passing LO1. (5 pts)

   Solution. See lecture notes.

   (b) Let $S = \{6, 11, 13, 22, 23, 26, 27, 46\}$ be an instance of `Set Partition (SP)`. Provide $f(S)$, where $f : \texttt{SP} \rightarrow \texttt{SS}$ is the mapping reduction from `SP` to `Subset Sum` provided in lecture. (10 pts)

   Solution. We have

   $$M = \sum_{s \in S} s = 174.$$

   Thus, $f(S) = (S, t = M/2 = 87)$.

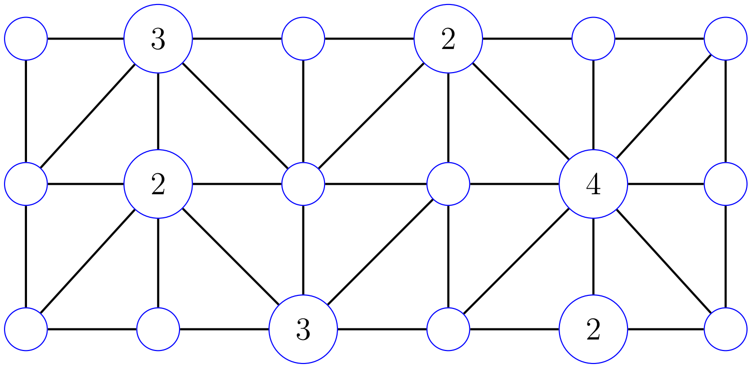
   (c) Using the problem instances in part b, verify that f maps a positive instance of `SP` to a positive instance of `SS`. (10 pts)

   Solution. $f(S) = (S, t = M/2 = 87)$ is positive for `Subset Sum` since $A = \{11, 23, 26, 27\}$ sums to 87. This means that A and $S - A$ forms a set partition since both sum to 87 and hence S is a positive instance of `SP`.

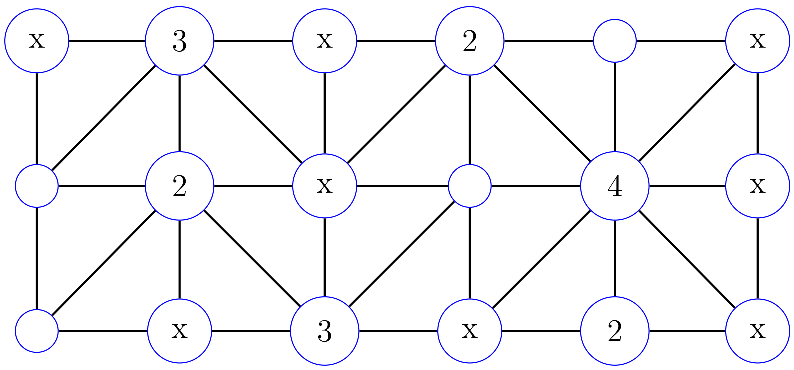
2. An instance of the `MineSweep` decision problem is a simple graph $G = (V, E)$ and a map $\text{val} : V \rightarrow \mathcal{N}$ for which some of G's vertices are labeled with a positive integer. Namely, if $\text{val}(v) = 0$, then v is not labeled, but if $\text{val}(v) > 0$, then $\text{val}(v)$ gives the value of the label. Note: correctly solving this problem counts for passing LO2.

   The problem is to decide if there is a way to place mines on some of G's unlabeled vertices so that, for each $v \in V$ that is labeled with some integer $k = \text{val}(v)$, exactly k of v's neighbors have been assigned a mine.

   (a) Show that the following labeled graph is a positive instance of `MineSweep`. Do so by placing an "x" in each node that should have a mine. (5 pts)



Solution.



(b) For a given instance $(G = (V, E), \text{val})$ of `MineSweep`, describe a certificate in relation to $(G = (V, E), \text{val})$. (5 pts)

Solution. We seek a map $\text{mine} : V \to \{0, 1\}$, where $\text{mine}(v) = 1$ iff a mine is placed at vertex $v \in V$.

(c) Provide a semi-formal verifier algorithm that takes as input i) an instance $(G = (V, E), \text{val})$ of `MineSweep`, and ii) a certificate for $(G = (V, E), \text{val})$ as defined in part a, and decides if the certificate is valid for $(G = (V, E), \text{val})$. (10 pts)

Solution.

For each $v \in V$,
  //Mines should not be placed on labeled nodes
  If $\text{val}(v) > 0 \wedge \text{mine}(v) = 1$, then return 0.
For each $v \in V$,
  If $\text{val}(v) = 0$, then continue.
  $\text{count} = 0$.
  For each $u \in V$,
    If $(u, v) \in E \wedge \text{mine}(u) = 1$, then $\text{count} = \text{count} + 1$.
  If $\text{count} \neq \text{val}(v)$, then return 0.
Return 1.

(d) Describe the running time of your verifier from part c using size parameters $m = |E|$ and $n = |V|$ and explain why it is a polynomial with respect to the size paramters.

Solution.

The verifier requires $O(n^2)$ steps due to the nested `for`-loops used the ensure that the value of each vertex corresponds with the number of its neighbors that have mines. Checking if a pair of vertices forms an edge can be done in constant time with the help of a hash table that requires $O(m) = O(n^2)$ steps to construct. Therefore, total number of steps is $O(n^2)$ which is a quadratic polynomial in n. (5 pts)

3. Recall the mapping reduction $f(\mathcal{C}) = (S, t)$, where f maps an instance of `3SAT` to an instance of the `Subset` decision problem. Given `3SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_2, x_3), (x_2, x_3, \overline{x}_4), (x_1, x_2, x_4), (\overline{x}_1, \overline{x}_3, \overline{x}_4)\}$$

answer the following questions about $f(\mathcal{C})$. Hint: to answer these questions you are *not* required to draw the table, but you might find it helpful. Note: scoring 18 or more points on this problem counts for passing LO3.

   (a) What is the value of t? (5 pts)

   Solution. $t = 11113333$ since $\mathcal{C}$ has $m = n = 4$ clauses and variables.

   (b) How many numbers (counting repeats) are in S? What is the largest (in terms of numerical value) number in S? (10 pts)

   Solution. In general, there are $2n+2m$ numbers (the y's and z's and g's and h's) . Hence $|S| = 8 + 8 = 16$. Largest number is $y_1 = 10001010$.

   (c) Determine a satisfying assignment for $\mathcal{C}$ and use it to identify a subset A of S that sums to t. List all the members of A. Hint: there are multiple possible answers, but the subset you choose must correspond with your chosen satisfying assignment. (10 pts)

   Solution. We have $\alpha = (x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1)$ satisfies $\mathcal{C}$. Therefore,

$$A = \{y_1, y_2, z_3, y_4, g_1, h_1, g_2, h_2, g_4, h_4\}$$

   sums to t.

4. Answer the following questions. Correctly answering at least two of the three is sufficient for passing LO4.

   (a) Provide the definition of what it means for a decision problem to be `NP`-complete. (6 pts)

   Solution. See lecture notes.

   (b) One of the exercises from the complexity lecture establishes a polynomial-time mapping reduction from `Independent Set` to `Vertex Cover`, i.e. `IS` $\leq_m^p$ `VC`. Using this reduction as the final link in the chain, provide the chain of polynomial-time mapping reductions that establishes the `NP`-completeness of `VC`, where each of the remaining reductions was provided in either the Mapping Reducibility or Complexity lecture. (7 pts)

   Solution. `SAT` to `3SAT`, `3SAT` to `Clique`, `Clique` to `Independent Set`, `Independent Set` to `Vertex Cover`

   (c) Each of the following decision problems described below takes as input a Boolean formula F. Classify each one as either being in `P`, `NP`, or co-`NP`. (4 pts each)

      i. F has at least two distinct satisfying assignments α_1 and α_2.
      Solution. `NP`. certificate: a pair of assignments α_1 and α_2

      ii. F is a **fallacy** meaning that it evaluates to 0 on every assignment α.
      Solution. `co-NP` (same as `UNSAT`)

iii. An instance of this problem has a second input $k > 0$ and the problem is to decide if F's Tseytin encoding (from `SAT` to `3SAT`) has at least k 3-SAT clauses.
Solution. `P`, since performing the Tseytin encoding takes polynomial time.

5. An instance $\mathcal{C}$ of the $\neq$-`SAT` decision problem is the same as that of `3SAT`, except now we seek a satisfying assignment α which is a **$\neq$-assignment**, meaning that, for each clause $c \in \mathcal{C}$, there must be a literal $l_1 \in c$ for which $\alpha(l_1) = 1$ *and* a literal $l_2 \in c$ for which $\alpha(l_2) = 0$. Solve the following.

(a) Find a $\neq$-assignment α that satisfies each of the following clauses.

$$\mathcal{C} = \{(\overline{x}_2, \overline{x}_3, \overline{x}_4), (\overline{x}_1, x_2, x_3), (x_1, x_3, \overline{x}_4), (\overline{x}_1, x_2, \overline{x}_4), (x_2, \overline{x}_3, \overline{x}_4)\}.$$

(5 pts)
Solution. $\alpha = (x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1)$ is a $\neq$-assignment (verify!).

(b) Given assignment α, $\overline{\alpha}$ is called the **complement** of α, where, for any literal l that is assigned by α,

$$\overline{\alpha}(l) = 1 - \alpha(l),$$

meaning that $\overline{\alpha}$ assigns each literal the complement value of what α assigns it. For the assignment α you provided in part a, verify that $\overline{\alpha}$ is also a $\neq$-assignment for the set of clauses in $\mathcal{C}$. In general, explain why the following statement is always true: "if α is a $\neq$-assignment for a set of clauses, then so is $\overline{\alpha}$". (5 points)

Solution. $\overline{\alpha} = (x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 0)$ is also a $\neq$-assignment (verify!). In general, if α is a $\neq$-assignment and $c = (l_1, l_2, l_3)$ is one of the clauses, then, if, say, α sets $l_1 = 1$ and $l_2 = 0$, then $\overline{\alpha}$ sets $l_1 = 0$ and $l_2 = 1$ and so $\overline{\alpha}$ is also a $\neq$-assignment with respect to c.

(c) Now consider the following mapping reduction from `3SAT` to $\neq$-`SAT`. Each clause $c \in \mathcal{C}$, where $c = (l_1, l_2, l_3)$, is mapped to the two clauses

$$f(c) = (l_1, l_2, z_c), (\overline{z_c}, l_3, g),$$

where z_c is a new Boolean variable that is specific to c, and g is a global Boolean variable that appears in the mapping of each c. Now suppose $\mathcal{C}$ is a negative instance of `3SAT`. We want to show that the above mapping can never result in a positive instance of $\neq$-`SAT`. To this end, let α be an assignment over the variables of $\mathcal{C}$ and suppose α does *not* satisfy $c = (l_1, l_2, l_3)$. How must z_c and g be assigned in order to make a $\neq$-assignment for $f(c)$? Explain. (5 pts)

Solution. We must have $z_c = 1$ and $g = 1$, since all other literals are assigned 0, and so both clauses need to have at least one literal assigned 1.

(d) Let β be the extension of α defined in the previous part. In other words, β assigns the variables in $\mathcal{C}$ in the same way as α, but it also assigns all the local z variables and the global g variable. Furthermore, suppose β is a $\neq$-assignment for $f(\mathcal{C})$. Then by part b, it must also be true that $\overline{\beta}$ is also a $\neq$-assignment for $f(\mathcal{C})$ and that it assigns the variables of $\mathcal{C}$ in the same manner as $\overline{\alpha}$. Moreover, since $\mathcal{C}$ is unsatisfiable, there must be a clause $c' = (l'_1, l'_2, l'_3)$ that is unsatisfied by $\overline{\alpha}$. Explain why this implies that $\overline{\beta}$ cannot form a $\neq$-assignment for the two clauses in $f(c')$, thus creating a contradiction. (10 pts)

Solution. Since each literal of c' is assigned 0 by $\overline{\beta}$, as was the case in part c, in order for $\overline{\beta}$ to be a $\neq$-assignment, $z_{c'}$ and g must both be assigned 1. But, $\beta(g) = 1$ by part c, and so $\overline{\beta}(g) = 0$ and so each literal of the clause $(\overline{z_{c'}}, l'_3, g)$ is assigned 0, which means $\overline{\beta}$ is *not* a $\neq$-assignment which contradicts the result from part b.

6. Recall that the `3-Dimensional Matching (3DM)` decision problem takes as input three sets A, B, and C, each having size n, along with a set S of triples of the form (a, b, c) where $a \in A$, $b \in B$, and $c \in C$. We assume that $|S| = m \geq n$. The problem is to decide if there exists a subset of n triples (called a **matching**) from S for which each member from $A \cup B \cup C$ belongs to exactly one of the triples.

   (a) Describe a valid mapping reduction f from `3DM` to `MineSweep` (See Problem 2 of this exam). Your description must be as general as possible so that it can be applied to any `3DM` instance. Also, you must argue that (A, B, C, S) is positive iff $f(A, B, C, S)$ is positive. Hint: make the instance $f(A, B, C, S)$ of `MineSweep` a bipartite graph with labels on one side. Clearly define the vertices, any labels on the vertices, and a rule for when two vertices are adjacent. (15 pts)

   Solution. $f(A, B, C, S) = G$ is a graph G whose vertices are $A \cup B \cup C \cup S$. Moreover for $u \in A \cup B \cup C$ and $v \in S$, (u, v) is an edge of G iff u appears in triple v. For example, since b appears in $(b, 2, y)$, there is an edge between these two vertices. Finally, we label all the vertices in $A \cup B \cup C$ with the number 1, meaning that exactly one edge incident with $u \in A \cup B \cup C$ must have a mine.

   Now suppose (A, B, C, S) is a positive instance of 3DM and let $M \subseteq S$ be a 3DM matching. Then by placing mines on each vertex $v \in M$, we see that, since M is a 3DM matching, every $u \in A \cup B \cup C$ will be adjacent to exactly one vertex having a mine, which implies G is a positive instance of `MineSweep`.

   Conversely, if $f(A, B, C, S) = G$ is a positive instance of `MineSweep`, then since there are $3n$ members of $A \cup B \cup C$, there must be n vertices of G which are members of S and that have mines placed on them. Call this set of vertices M. Moreover, for every $u \in A \cup B \cup C$ there must be exactly one triple in M for which u appears in that triple. In other words, M must be a 3DM matching for (A, B, C, S), and so (A, B, C, S) is a positive instance of 3DM.

   (b) Demonstrate the reduction from part a using the `3DM` instance (A, B, C, S), where $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y, z\}$, and

   $$S = \{(b, 2, y), (b, 1, z), (a, 3, z), (c, 2, y), (a, 2, y), (a, 3, y), (c, 3, x), (c, 1, z), (b, 1, x)\}.$$

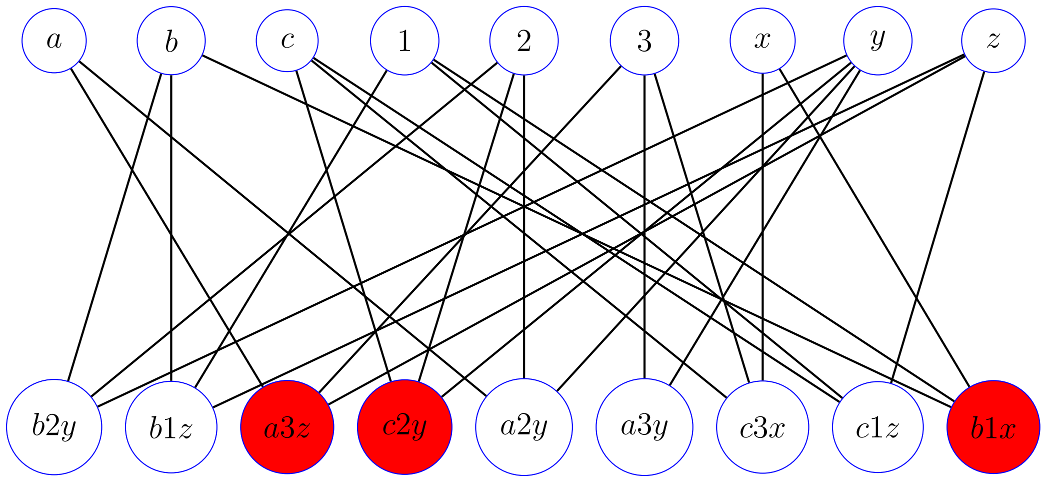
   (10 pts)

   Solution.

   Below is a drawing of the graph $G = f(A, B, C, S)$, where $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x, y, z\}$, and

   $$S = \{(b, 2, y), (b, 1, z), (a, 3, z), (c, 2, y), (a, 2, y), (a, 3, y), (c, 3, x), (c, 1, z), (b, 1, x)\}.$$

   Since $M = \{(a, 3, z), (b, 1, x), (c, 2, y)\}$ is a 3DM matching for (A, B, C, S), we see that (A, B, C, S) is a positive instance of 3DM, and that placing mines at these vertices establishes that G is a positive instance of `MineSweep`.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