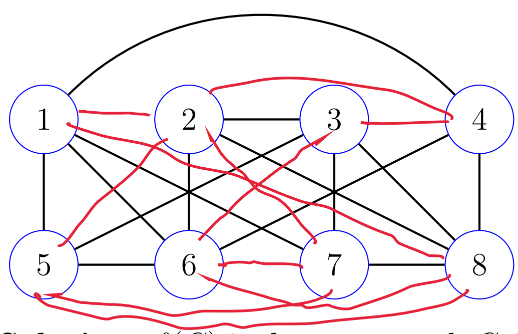
CECS 329, LO Assessment 7, March 21st, Spring 2024, Dr. Ebert

Problems

LO3. Solve the following.

(a) Suppose the graph G shown below is an instance of the `Hamilton Cycle` decision problem. Draw $f(G)$, where f is the mapping reduction from `HC` to the `Traveling Salesperson` decision problem. Hint: remember the k value!



red edges: weight = 8 = |V| = K

black edges: weight = 1

Solution. $f(G)$ is the same graph G, but with added edges and weights drawn in red (see above). Here, $k = |V| = 8$ is the target maximum tour cost.

(b) Verify that the mapping is valid by verifying that G and $f(G)$ are either both positive or both negative instances. Defend your answer.

Solution. G is a positive instance of `HC` via $C = 1, 4, 8, 7, 3, 2, 6, 5, 1$. Moreover, $f(G)$ is positive for `TSP` since C is a cycle in the $f(G)$ graph and that has a total cost equal to $8 \leq k = 8$.

LO4. Answer the following. Note: correctly answering two of the three constitutes a pass.

(a) Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

i. An instance of the `Vertex Cover` decision problem is a pair (G, k), where $G = (V, E)$ is a simple graph, $k \geq 0$ is an integer, and the problem is to decide if G has a **vertex cover** of size k, i.e. a set $C \subseteq V$ for which every edge $e \in E$ is incident with at least one vertex in C.

ii. An instance of `Substring` is a pair (s_1, s_2) of binary strings and the problem is to decide if s_1 occurs as a substring of s_2. For example $(10101, 001010111)$ is a positive instance of `Substring` since 10101 is a substring of 001010111.

iii. An instance of `Bounded Independent Set` is a graph $G = (V, E)$ and an integer $k \geq 0$ and the problem is to decide if the maximum independent set in G is of a size that does not exceed k.

Solution. i: `NP`, ii: `P`, iii: `co-NP`

(b) Describe the three main steps that must be completed in order to establish that a decision problem L is a member of `NP`. Clearly define all technical terms. Hint: your three steps should make reference to two different technical terms that need defining.

Solution. See Complexity lecture notes.

(c) Provide the chain of polynomial-time mapping reductions provided in lecture that establishes the `NP`-completeness of `TSP`. Hint:

$$\texttt{Undirected Hamilton Path (HP)} \leq_m^p \texttt{Hamilton Cycle (HC)}$$

is one of the reductions.

Solution.

$$\texttt{SAT} \leq_m^p \texttt{3SAT} \leq_m^p \texttt{DHP} \leq_m^p \texttt{UHP} \leq_m^p \texttt{HC} \leq_m^p \texttt{TSP}.$$

LO5. Solve the following.

(a) Provide the instructions of a URM program that computes the function $f(x) = 2^x$.

(b) Describe the role each register plays in computing $f(x)$.

Solution. Idea of program: compute the first $x+1$ powers of 2: $2^0, 2^1, \ldots, 2^x$.

R_2: Store current power of 2

R_3: Counts # of completed powers of 2

R_4: Counts up to current power of 2

R_5: Counts x2 up to current power of 2

1. S(2)
2. J(1, 3, 13)
3. Z(4)
4. Z(5)
5. J(2, 4, 10)
6. S(4)
7. S(5)
8. S(5)
9. J(1, 1, 5)
10. S(3)
11. T(5, 2)
12. J(1, 1, 2)
13. T(2, 1)

LO6. Solve the following problems.

(a) Provide the URM program P whose Gödel number equals

$$2^{55} + 2^{73} + 2^{114} + 2^{133} - 1.$$

Show all work. $B(I_1) = 55$ $\quad B(I_2) = 73 - 55 - 1 = 17$

Solution. $B(I_3) = 114 - 73 - 1 = 40$ $\quad B(I_4) = 133 - 114 - 1 = 18$

$55 \bmod 4 = 3 \Rightarrow$ Jump $\quad (55-3)/4 = 13 \quad 13+1 = 14 = 2^1((2)(\boxed{3})+1)$

$1+1 = 2 = 2^{\boxed{1}}(2(\boxed{0})+1) \Rightarrow J(2,1,4)$

$17 \bmod 4 = 1 \Rightarrow$ Successor $\quad (17-1)/4 = 4 \Rightarrow S(5)$

$40 \bmod 4 = 0 \Rightarrow$ Zero $\quad 40/4 = 10 \Rightarrow Z(11)$

$18 \bmod 4 = 2 \Rightarrow$ Transfer $\quad 16/4 = 4 \quad 4+1 = 5 = 2^{\boxed{0}}((2)(\boxed{2})+1) \Rightarrow T(1,3)$

$\therefore P = J(2,1,4), S(5), Z(11), T(1,3)$

(b) A universal program P_U is simulating a program that has 286 instructions and whose Gödel number is

$$x = 2^3 + 2^{179} + 2^{191} + 2^{215} + 2^{223} + 2^{275} + \cdots + 2^{c_{286}} - 1.$$

If the current configuration of the computation of P_x on some input has encoding

$\tau(C_{next}) = 2^3 + 2^7 + 2^{13} + 2^{20} - 1$

$$\sigma = 2^3 + 2^7 + 2^{13} + 2^{16} - 1,$$

Thus $I_2 = J(1,2,6) \Rightarrow C_{next} = (3,3,5,6)$

then provide the next configuration of the computation *and* its encoding.

Solution.

$C_{cur} = (3, 7-3-1, 13-7-1, 16-13-1) =$

$(3, 3, 5, 2)$

$B(I_2) = 179 - 3 - 1 = 175 \quad 175 \bmod 4 = 3 \Rightarrow J(i,j,k)$

where $\xi(i-1, j-1, k-1) = \left(\frac{175-3}{4}\right) = 43$

$43 + 1 = 44 = 2^2((2)(\boxed{5})+1) \Rightarrow k = 5+1 = 6$ Also, $2+1 = 3 = 2^{\boxed{0}}((2)(\boxed{1})+1)$

$\Rightarrow i = 0+1$ and $j = 1+1 = 2$.

LO7. Answer the following. Note: correctly answering two of the three constitutes a pass.

(a) Describe what it means to be a positive instance of the `Total` decision problem.

Solution. See Undecidability and Diagonalization lecture notes.

(b) In applying the diagonalization method towards proving the undecidability of `Total`, we defined a computable function $g(x)$ in terms of $f(x)$, the decision function for `Total`, which we assume to be total computable. Provide the formula for computing $g(x)$ and describe what needs to be established about $g(x)$ in order for the diagonalization method to be successful.

Solution. See Undecidability and Diagonalization lecture notes.

$$g(x) = \begin{cases} \phi_x(x) + 1 & \text{if } f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Suppose URM program P_{51} computes $\phi_{51}(x)$ and that

$f(51) = 0$

$$\phi_{51}(x) = \begin{cases} \lfloor x/2 \rfloor & \text{if } x \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

What is the value of $g(51)$? Defend your answer.

Solution. Since ϕ_{51} is not a total computable function, $g(51) = 0$. Notice that $g(51) = 0 \neq \phi_{51}(51) =\uparrow$ which ensures that $g \neq \phi_{51}$.