CECS 329, Learning Outcome Assessment 6, March 14th, Spring 2024, Dr. Ebert

Problems

LO2. An instance of the `Max Cut` decision problem is a simple graph $G = (V, E)$ and an integer $k \geq 0$. The problem is to decide if there is a way to color the vertices of G using the colors red and blue and results in there being at least k edges $e = (u, v)$ for which u and v are assigned different colors.

(a) For a given instance (G, k) of `Max Cut`, describe a certificate in relation to (G, k).

Solution. R is a subset of vertices to color red.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k) of `Max Cut`, ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for (G, k).

Solution.

Count = 0
For each $e = (u, v) \in E$
If $(u \in R \oplus v \in R) = 1$, then Count++.
Return (Count $\geq k$).

(c) Provide size parameters that may be used to measure the size of an instance (G, k) of `Max Cut`.

Solution. $m = |E|$ $n = |V|$

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

The verifier runs in $O(m)$ steps so long as we store the edges in a table, which would require $O(m)$ steps to create.

LO3. An instance $\mathcal{C}$ of 3SAT consists of clauses $c_1 = (x_1, \overline{x}_2, x_3)$, $c_2 = (\overline{x}_2, x_3, x_4)$, $c_3 = (\overline{x}_1, x_2, \overline{x}_4)$, and $c_4 = (\overline{x}_1, \overline{x}_3, x_4)$. Answer the following questions about the mapping reduction $f(\mathcal{C}) = (G, k)$ provided in lecture from 3SAT to Clique and applied to instance $\mathcal{C}$.

(a) How many vertices and edges does G have? Explain and show work. Hint: there are six different vertex-group pairs.

Solution. $3 \cdot 4 = 12$ vertices

$3 = |\mathcal{C}|$

C_1-C_2 edges: 9 C_1-C_3 edges: 9-2 C_1-C_4 edges: 9-2

C_2-C_3 edges: 9-2 C_2-C_4 edges: 9-1 C_3-C_4 edges: 9-1

Total Edges = 46

(b) What is the value of k?

Solution.

$k = 4$ since a clique must include one vertex from each of the 4 clause groups.

(c) Given that $\alpha = (x_1 = x_2 = 0, x_3 = 1, x_4 = 0)$ satisfies $\mathcal{C}$, provide a clique set for G that certifies (G, k) is a positive instance of Clique. Hint: for each clique member, indicate the group from which it came.

Solution.

Clique for G: $\{\overline{x}_2, \overline{x}_2, \overline{x}_1, \overline{x}_1\}$

$C_1 \quad C_2 \quad C_3 \quad C_4$

All literals are pairwise consistent and so form a 4-clique in G.

LO4. Answer the following questions. Correctly answering at least two of the three is sufficient for passing

(a) Provide the definition of what it means for a decision problem to be NP-complete.

Solution. See Complexity lecture notes.

(b) Describe the three main steps that must be completed in order to establish that a decision problem L is a member of NP. Clearly define all technical terms.

Solution.

i. Define a certificate set for the problem instance which is a set of potential solutions for the instance.

ii. Provide a verifier program that takes two inputs, a certificate c and problem instance x, and decides if c is valid (i.e. a solution) for x.

iii. Prove that the verifier requires a polynomial number of steps in terms of the size of the problem instance.

(c) Each of the following graph decision problems described below takes as input a simple graph $G = (V, E)$ and a nonnegative integer $k \geq 0$. Classify each one as either being in P, NP, or co-NP.

i. Decide if G has fewer than k connected components.
ii. Decide if the size of every independent set of G is less than or equal to k.
iii. Decide if G has a vertex cover of size k.

Solution. i: P, ii. co-NP, iii. NP

LO5. Provide the instructions of a URM program that computes the function $f(x) = x^2$. For each register used, provide a few sentences describing how it is used.

Solution.

$$x^2 = \underbrace{x + x + x \cdots + x}_{x\text{-times}}$$

```
1:J(1,2,9)
2:J(1,3,6)
3:S(3) //Count up to x
4:S(4) //Count up to x^2
5:J(1,1,2)
6:Z(3)
7:S(2) //Count the number of times R3 has counted up to x
8:J(1,1,1)
9:T(4,1)
```

LO6. Solve the following problems.

(a) Provide the URM program P whose Gödel number equals

$$2^{22} + 2^{35} + 2^{65} + 2^{133} - 1.$$

Show all work.

$I_1 =$

Solution.

$\beta(I_1) = 22$ and $22 \bmod 4 = 2 \Rightarrow T(i,j)$,

where $\pi(i-1, j-1) = \frac{20}{4} = 5$

$5 + 1 = 6 = 2^1((2)(1) + 1) \Rightarrow i = 1 + 1 = 2$ and $j = 1 + 1 = 2$

So, $T(2,2) = I_1$

$\beta(I_2) = 35 - 22 - 1 = 12$ and $12 \bmod 4 = 0 \Rightarrow Z(i) = I_2$

where $i = 4$. So, $I_2 = Z(4)$.

$\beta(I_3) = 65 - 35 - 1 = 29$ and $29 \bmod 4 = 1 \Rightarrow I_3 = S(i)$

where $i = 8$. So $I_3 = S(8)$

$\beta(I_4) = 133 - 65 - 1 = 67$ and $67 \bmod 4 = 3 \Rightarrow J(i,j,k)$

where $\xi(i-1, j-1, k-1) = 16$. $16 + 1 = 17 = 2^0((2)(8) + 1) \Rightarrow$

$i = 1, j = 1, k = 9$. So, $I_4 = J(1,1,9)$

$\therefore P = T(2,2), Z(4), S(8), J(1,1,9)$

(b) A universal program P_U is simulating the computation of a program P_x on some input y, where P_x that has 136 instructions and whose Gödel number is

$$x = 2^7 + 2^{31} + 2^{43} + 2^{55} + \underline{2^{65}} + 2^{109} + \cdots + 2^{c_{136}} - 1.$$

If the current configuration of the computation has encoding $\beta(I_5) = 9$

$$\sigma = 2^5 + 2^{10} + 2^{18} + 2^{24} - 1,$$

then provide the next configuration of the computation *and* its encoding.

Solution.

$C_0 = \tau^{-1}(\sigma) = (5, 4, 7, \underset{PC}{\underset{\shortparallel}{5}})$

$\beta(I_5) = 65 - 55 - 1 = 9 \Rightarrow I_4 = S(3).$

Thus, $C_{next} = (5, 4, 8, 6)$ and

$\tau(C_{next}) = 2^5 + 2^{10} + 2^{19} + 2^{26} - 1$