CECS 329, Learning Outcome Assessment 5, March 7th, Fall 2024, Dr. Ebert

Problems

LO1. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B.

Solution. See Mapping Reducibility lecture notes for the definition.

(b) Recall the mapping reduction $f : \texttt{HP} \to \texttt{LPath}$ from the `Hamilton Path` decision problem to the `LPath` decision problem. Suppose Karla has a graph G with 84 vertices and 482 edges and, as part of her online-currency-trading algorithm, must determine if G has a Hamilton path. Her friend Brian has implemented the function

```
Boolean has_path(Graph G, int k);
```

which returns 1 iff input G has a simple path of length k. Explain how Karla can use Brian's function to solve her problem. In particular, what arguments must she pass into his function and why will his function give her the correct answer?

Solution. Karla's function call must be

```
[redacted] has_path(G,83);
```

since G has a Hamilton path iff it has a simple path of length 83.

LO2. An instance $(\mathcal{C}, m)$ of `Set Splitting` is a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_n\}$, where, for each $i = 1, \ldots, n$, $C_i \subseteq \{1, 2, \ldots, m\}$. The problem is to decide if there is a set $A \subseteq \{1, 2, \ldots, m\}$ for which, for every $i = 1, \ldots, n$, both $A \cap C_i \neq \emptyset$ and $\overline{A} \cap C_i \neq \emptyset$, where $\overline{A} = \{1, 2, \ldots, m\} - A$.

(a) For a given instance $(\mathcal{C}, m)$ of `Set Splitting` describe a certificate in relation to $(\mathcal{C}, m)$.

Solution. Certificate: Subset $A \in \{1, \ldots, m\}$

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance $(\mathcal{C}, m)$ of `Set Splitting`, ii) a certificate for $(\mathcal{C}, m)$ as defined in part a, and decides if the certificate is valid for $(\mathcal{C}, m)$.

Solution.

For each $i = 1, \ldots, n$
  inA = 0
  outA = 0
  For each $j \in C_i$
    If $j \in A$, inA = 1
    Else outA = 1
  If inA = 0 ∨ outA = 0, return 0
Return 1.

(c) Provide size parameters that may be used to measure the size of an instance $(\mathcal{C}, m)$ of `Set Splitting`.

Solution. $m, n = |\mathcal{C}|$

The outer loop makes $O(n)$ iterations, while inner loop makes $O(m)$ iterations. Moreover, each step within the nested loop requires $O(1)$ steps since we may represent each C_i and A as arrays.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

Solution. ∴ Running time is $O(mn)$.

LO3. Recall the mapping reduction $f(\mathcal{C}) = (S, t)$, where f maps an instance of `3SAT` to an instance of the `Subset Sum` decision problem. Given `3SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_2, \overline{x}_3), (\overline{x}_1, x_3, \overline{x}_4), (x_1, \overline{x}_3, x_4), (\overline{x}_1, \overline{x}_3, x_4), (\overline{x}_2, x_3, \overline{x}_4)\}$$

answer the following questions about $f(\mathcal{C})$.

(a) What is the value of t?

Solution. $t = \underbrace{1111}_{4 \text{ times}}\underbrace{33333}_{5 \text{ times}}$

$n = |V| = 4$

$m = |\mathcal{C}| = 5$

(b) How many numbers (counting repeats) are in S? What is the largest (in terms of numerical value) number in S?

Solution. $|S| = 2(n+m) = 2(9) = 18$

Largest: $y_1 = 100{,}010{,}100$

(c) Determine a satisfying assignment for $\mathcal{C}$ and use it to identify a subset A of S that sums to t. List all the members of A. Hint: there are multiple possible answers, but the subset you choose must correspond with your chosen satisfying assignment.

Solution. $\alpha = (x_1 = x_2 = x_3 = x_4 = 0)$

$A = \{z_1, z_2, z_3, z_4, g_1, g_2, g_3, h_3, g_4, g_5\}$

LO4. Answer the following questions. Correctly answering at least two of the three is sufficient for passing

(a) Provide the definition of what it means for a decision problem to be NP-complete.
Solution. See Complexity lecture notes.

(b) Describe the three main steps that must be completed in order to establish that a decision problem L is a member of NP. Clearly define all technical terms. **Solution.** See Complexity lecture notes.

(c) Each of the following decision problems described below takes as input a Boolean formula F. Classify each one as either being in P, NP, or co-NP.

I. Decide if F is a **tautology**, meaning that it evaluates to 1 on every assignment α.

II. Decide if F has at least k $\wedge$-nodes in its parse tree, where $k \geq 0$ is a second input to the decision problem.

III. Decide if there is an assignment α over the variables of F for which $F(\alpha) = F(\overline{\alpha})$, where $\overline{\alpha}(x) = 1 - \alpha(x)$, for every variable x of F.

Solution. I: co-NP, II: P, III: NP

LO5. Provide the instructions of a URM program that computes the function $f(x, y) = x/y$. For each register used, provide a few sentences describing how it is used. Hint: you may assume $y > 0$ and that any fractional portion of x/y is truncated. For example, $5/2 = 2$, while $18/5 = 3$.

```
R3: count number of times y has been counted
R4: up to x
R5: count up to y

1: J(1,4,10)
2: Z(5)
3: J(2,5,8)
4: J(1,4,10)
5: S(5)
6: S(4)
7: J(1,1,3)
8: S(3)
9: J(1,1,1)
10: T(3,1)
```