**CECS 329, Learning Outcome Assessment 10, April 20th, Spring 2023, Dr. Ebert**

**NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit solutions to at most 2 LO problems on separate sheets of paper.**

# Problems

LO6. Do the following.

(a) Compute the Gödel number for program $P = T(2,3), Z(4), J(1,2,4), S(1)$. Write your answer as a sum of powers of two minus 1.

**Solution.**

$$\beta(T(2,3)) = 4\pi(1,2) + 2 = 4(9) + 2 = 38.$$

$$\beta(Z(4)) = 4(3) = 12.$$

$$\beta(J(1,2,4)) = 4\xi(0,1,3) + 3 = 4\pi(\pi(0,1),3) + 3 = 4\pi(2,3) + 3 = 4(27) + 3 = 111.$$

$$\beta(S(1)) = 4(0) + 1 = 1.$$

$$\gamma(P) = \tau(38, 12, 111, 1) = 2^{38} + 2^{51} + 2^{153} + 2^{155} - 1.$$

(b) Provide the URM program $P$ whose Gödel number equals

$$2^{79} + 2^{89} + 2^{104} + 2^{121} - 1.$$

**Solution.** $\gamma^{-1}(x) = J(1,2,3), S(3), T(3,1), Z(5)$.

LO7. Answer and solve the following.

(a) Which of the following is *not* needed by a universal program $P_U$ in order to simulate the computation of program $P_x$ on input $y$?

i. the maximum index of any register used by $P_x$
ii. **X**the maximum number of configurations used in the computation of $P_x$ on input $y$
iii. the number of instructions of $P_x$
iv. the length of each configuration used in the computation of $P_x$ on input $y$

(b) A universal program $P_U$ is simulating a program that has 253 instructions and whose Gödel number is

$$x = 2^{79} + 2^{102} + 2^{120} + 2^{129} + 2^{c_5} + \cdots + 2^{c_{253}} - 1.$$

If the current configuration of the computation of $P_x$ on some input has encoding

$$\sigma = 2^2 + 2^4 + 2^{10} + 2^{19} + 2^{24} - 1,$$

then provide the next configuration of the computation *and* its $\tau$ encoding.

**Solution.**

Current configuration: $\tau^{-1}(\sigma) = (2, 1, 5, 8, 4)$, Program counter equals 4, and $\beta(I_4) = 129 - 120 - 1 = 8$ which implies a `Zero` instruction, since $8 \bmod 4 = 0$. Moreover, $8/4 = 2$, which yields $I_4 = Z(3)$. Finally, the next configuration is

$$(2, 1, 0, 8, 5),$$

and

$$\sigma_{\text{next}} = \tau(2, 1, 0, 8, 5) = 2^2 + 2^4 + 2^5 + 2^{14} + 2^{20} - 1.$$

LO8. Do the following.

(a) In one or more complete sentences, describe what is asserted by the Church-Turing Thesis.

**Solution.** Any [functional] deterministic step-by-step procedure that can be executed by a human using pencil and paper can be represented by a URM program.

(b) Consider the function

$$g(x) = \begin{cases} 1 & \text{if } \phi_x \text{ is total} \\ 0 & \text{otherwise} \end{cases}$$

In other words, $g(x) = 1$ iff program $P_x$ halts on all of its inputs. We want to prove that $g(x)$ is undecidable, meaning there is no URM program that computes $g$. To do this, let's assume that $g(x)$ is computable by some URM program $G$. Then define the function $f(x)$ as follows.

$$f(x) = \begin{cases} P_x(x) + 1 & \text{if } g(x) = 1 \\ 0 & \text{if } g(x) = 0 \end{cases}$$

Provide an informal description of the steps you would use to compute $f(x)$ using pencil and paper. Why does this imply that $f(x)$ is URM computable?

**Solution.** Simulate program $G$ on input $x$ to obtain $g(x)$. If the output is 0, then return 0. If the output is 1, then simulate program $P_x$ on input $x$ and return $P_x(x) + 1$.

(c) Let $e$ denote the Gödel number of the URM program $F$ that computes $f(x)$ from part b. In other words $F = P_e$. Show that a contradiction arises when we try to compute $f(e)$ using $F$.

**Solution.** Note that $f$ is total (why?) so, by definition $f(e) = F(e) = P_e(e)$. But by the rule defined above for $f$, we have $f(e) = P_e(e) + 1 = F(e) + 1$, a contradiction.

LO9. An instance of the decision problem `Odd Range` is a Gödel number $x$, and the problem is to decide if function $\phi_x$ has a range that consists only of odd natural numbers. Mathematically speaking, $g(x) = 1$ iff $E_x \subseteq \{1, 3, 5, \ldots\}$. Consider the function

$$g(x) = \begin{cases} 1 & \text{if } E_x \subseteq \{1, 3, 5, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Evaluate $g(x)$ for each of the following Gödel number's $x$. Note: 2 out of 3 correct is considered passing. **Justify your answers**.

i. $x = e_1$, where $e_1$ is the Gödel number of the program that computes the function $\phi_{e_1}(y) = y^3$.
**Solution.** $g(e_1) = 0$ since $\phi_{e_1}(2) = 2^3 = 8$ is not odd.

ii. $x = e_2$, where $e_2$ is the Gödel number of the program that computes the function $\phi_{e_2}(y) = 3^y$.
**Solution.** $g(e_2) = 1$ since $\phi_{e_2}$ has range

$$\{3^0, 3^1, 3^2, 3^3, 3^4, \ldots\} = \{1, 3, 9, 27, 81, \ldots\}$$

which is a set of odd numbers since odd $\times$ odd $=$ odd.

iii. $x = e_3$, where $e_3$ is the Gödel number of the program that computes $g(x)$ (assuming that $g(x)$ is URM computable).
**Solution.** $g(e_3) = 0$ since $g(x)$ sometimes must output 0 (see problem i) which is even.

(b) Prove that $g(x)$ is not URM computable. In other words, there is no URM program that, on input $x$, always halts and either outputs 1 or 0, depending on whether or not $\phi_x$ has a range consisting of odd natural numbers. Do this by writing a program $P$ that uses $g$ and makes use of the `self` programming concept. Then show how $P$ creates a contradiction.

**Solution.** Consider the following program $P$.

Input $y \in \mathcal{N}$.
If $g(\texttt{self}) = 1$, Return 0.
Return 1.

Let $e$ denote $P$'s Gödel number. If $g(e) = 1$, then it means $P$ has an odd range, but in this case the program returns 0 for each input $y$, which implies $E_e = \{0\}$ which is not a set of odd numbers, a contradiction.

If $g(e) = 0$, then it means $P$ does not have an odd range, but in this case the program returns 1 on each input $y$, which implies $E_e = \{1\}$, which is a set of odd numbers.

Therefore, $g$ cannot be computable and hence the `Odd Range` decision problem is undecidable.

LO10. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from decision problem $A$ to decision problem $B$. (See Page 4 of Turing and Map Reducibility Lecture).

(b) For the mapping reduction $f$ : `Subset Sum` $\rightarrow$ `Set Partition`, determine $f(S, t)$ for `Subset Sum` instance ($S = \{2, 10, 14, 21, 33, 38, 46\}, t = 61$). Verify that $(S, t)$ and $f(S, t)$ are both positive instances of their respective decision problems. Show work.

**Solution.** The members of $S$ sum to $M = 164$, and so we must add $J = M - 2t = 164 - 122 = 42$ to $S$ in order to get $S'$:

$$f(S, t) = S' = S \cup \{42\}.$$

$(S, t)$ is a positive instance of `Subset Sum` since $2 + 21 + 38 = 61$. Also, $S' = S \cup \{42\}$ is a positive instance of `Set Partition` since $A = \{2, 21, 38, 42\}$ and $B = \{10, 14, 33, 46\}$ is a partition of $S'$ and both sets sum to 103 (verify).