# CECS 528, Learning Outcome Assessment 9, Yellow, Fall 2023, Dr. Ebert

**NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper**.

## Problems

LO5. Answer the following with regards to a correctness-proof outline for the Fractional Knapsack algorithm.

(a) Assume $x_1, x_2, \ldots, x_n$ is an ordering of the items in *decreasing* order of profit density (i.e. profit per unit weight). Let $f_i \in [0, 1]$ denote the fraction of item $x_i$ that the FK-algorithm adds to the knapsack, $i = 1, 2, \ldots, n$. Explain why $f_1 \geq f_2 \geq \cdots \geq f_n$ is a non-increasing sequence of fractions.

(b) Let $f'_1, f'_2, \ldots, f'_n$ be a sequence of fractions that optimizes total profit, and assume that $f_i = f'_i$, for all $i < k$, but $f_k \neq f'_k$. Explain why, in this case, it must be true that $f'_k < f_k$. Hint: what is the contradiction in case the opposite was true?

(c) From part b, the optimal solution uses $(f_k - f'_k)w_k$ less weight of item $x_k$. Suppose it uses $(f_k - f'_k)w_k$ *more* weight of item $x_{k+1}$ than does FKA. Show that the FKA solution will earn at least as much profit on items $x_1, \ldots, x_k, x_{k+1}$ as the optimal solution will earn on these same items. In other words, show that the difference between the FKA total profit and the optimal total profit is nonnegative. Why does this imply that both total profits are equal?

LO6. For the weighted graph with edges

$$(a, b, 5), (b, d, 3), (c, e, 6), (c, f, 2), (d, e, 4), (d, f, 1),$$

Show how the membership-tree forest (not the Kruskal forest!) changes when processing each edge in the Kruskal sorted order when performing Kruskal's algorithm. When merging two trees, use the convention that the root of the merged tree should be the one having *lower* alphabetical order. For example, if two trees, one with root $a$, the other with root $b$, are merged, then the merged tree should have root $a$.

LO7. Answer the following.

(a) Provide the dynamic-programming recurrence for computing the maximum-cost path, denoted $\text{mc}(u, v)$, from a vertex $u$ to a vertex $v$ in a directed acyclic graph (DAG) $G = (V, E, c)$, where $c(x, y)$ gives the cost of edge $e = (x, y)$, for each $e \in E$. The recurrence should allow one to compute the maximum costs from a single source to all other vertices in a linear number of steps. Hint: step backward from $v$.

(b) Draw the vertices of the following DAG $G$ in a linear left-to-right manner so that the vertices are topologically sorted, meaning, if $(u, v)$ is an edge of $G$, then $u$ appears to the left of $v$. The vertices of $G$ are a-h, while the weighted edges of $G$ are

$$(a, b, 1), (a, e, 2), (a, f, 3), (b, c, 3), (b, g, 2), (c, d, 4), (c, g, 2), (c, h, 5), (d, h, 2), (e, b, 3), (e, f, 4),$$

$$(f, b, 4), (f, c, 3), (f, g, 1), (g, d, 5), (g, h, 2).$$

(c) Starting from left to right in topological order, use the recurrence to compute

$$\text{mc}(a, a), \ldots, \text{mc}(a, h).$$

LO8. Do/answer the following.

(a) Draw the implication graph $G_{\mathcal{C}}$ associated with the `2SAT` instance

$$\mathcal{C} = \{(\overline{x}_1, x_2), (\overline{x}_1, x_3), (\overline{x}_2, x_3), (x_2, x_4), (\overline{x}_3, x_4), (\overline{x}_3, \overline{x}_4)\}.$$

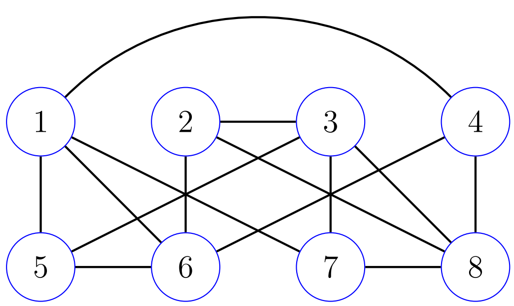
(b) Apply the improved `2SAT` algorithm to obtain a satisfying assignment for $\mathcal{C}$. When deciding on the next reachability set $R_l$ to compute, follow the literal order $l = x_1, \overline{x}_1, \ldots, x_4, \overline{x}_4$. For each consistent reachability set encountered, provide the partial assignment $\alpha_{R_l}$ associated with $R_l$ and draw the reduced implication graph before continuing to the next reachability set. Note: do *not* compute the reachability set for a literal that has already been assigned a truth value. Provide a final assignment $\alpha$ and verify that it satisfies all six clauses.

(c) Suppose a `Reachability`-oracle answers "yes" to the query reachable$(G_{\mathcal{C}}, \overline{x}_2, x_2)$. If $\mathcal{C}$ is satisfiable via assignment $\alpha$, then what is the value of $\alpha(x_2)$? Explain.

LO9. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction fromm problem $A$ to problem $B$.

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Maximum Independent Set (MIS)` optimization problem. Draw $f(G)$, where $f$ is the mapping reduction from `MIS` to `Maximum Clique` provided in lecture.




(c) Verify that $f$ is valid for input $G$ in the sense that both $G$ and $f(G)$ have the same solution. Hint: for both problems please assume that a "solution" is represented by a subset of vertices.