CECS 528, Learning Outcome Assessment 9, Pink, Fall 2023, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems

LO5. Answer the following with regards to a correctness-proof outline for Prim's algorithm.

(a) In the correctness proof of Prim's algorithm, suppose $T = e_1, \ldots, e_{n-1}$ are the edges selected by Prim's algorithm (in that order) and T_{opt} is an mst that uses edges $e_1, \ldots, e_{k-1}$, but for which $e_k \notin T_{\text{opt}}$. Explain why e_k is incident with one vertex in T_{k-1} (Prim's tree after round $k-1$) and with one vertex *not* in T_{k-1}. Hint: your answer should have *nothing* to do with the fact that $e_k \notin T_{\text{opt}}$.

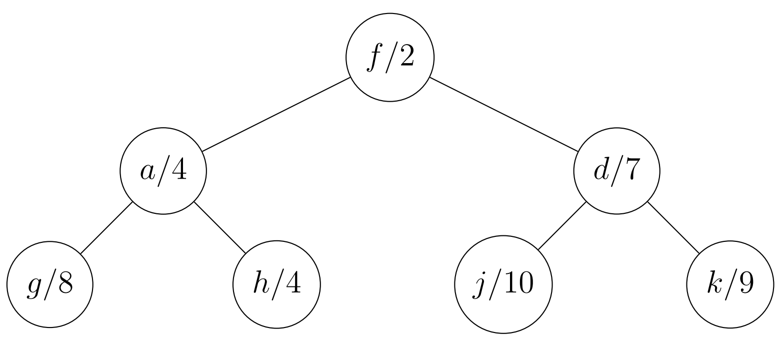
(b) Since $e_k \notin T_{\text{opt}}$, it follows that $T_{\text{opt}} + e_k$ has a cycle C. Explain why there must be an edge $e \in C$ for which i) $e \neq e_k$ and ii) e is incident with one vertex in T_{k-1} and with one vertex *not* in T_{k-1}. Furthermore, explain why $w(e) \geq w(e_k)$.

(c) Explain why $T_{\text{opt}} - e + e_k$ is also an mst, i.e. a tree of minimum cost.

LO6. The tree below shows the state of the binary min-heap at the beginning of some round of Prim's algorithm, applied to some weighted graph G. If G has edges

$$(a, f, 6), (d, g, 4), (f, g, 10), (f, h, 2), (f, k, 3),$$

then draw a plausible state of the heap at the end of the round.



LO7. Answer the following.

(a) Provide the dynamic-programming recurrence for computing the distance $\mathrm{d}(u, v)$, from a vertex u to a vertex v in a directed acyclic graph (DAG) $G = (V, E, c)$, where $c(x, y)$ gives the cost of edge $e = (x, y)$, for each $e \in E$. The recurrence should allow one to compute the distance from a single source to all other vertices in a linear number of steps. Hint: step backward from v.

(b) Draw the vertices of the following DAG G in a linear left-to-right manner so that the vertices are topologically sorted, meaning, if (u, v) is an edge of G, then u appears to the left of v. The vertices of G are a-h, while the weighted edges of G are

$$(a,b,1),(a,e,2),(a,f,3),(b,c,3),(b,g,2),(c,d,4),(c,g,2),(c,h,5),(d,h,2),(e,b,3),(e,f,4),$$

$$(f,b,4),(f,c,3),(f,g,1),(g,d,5),(g,h,2).$$

(c) Starting from left to right in topological order, use the recurrence to compute

$$d(a,a),\ldots,d(a,h).$$

LO8. Do/answer the following.

(a) Draw the implication graph $G_{\mathcal{C}}$ associated with the `2SAT` instance

$$\mathcal{C} = \{(\overline{x}_1, x_2), (\overline{x}_1, x_3), (\overline{x}_2, x_3), (x_2, x_4), (\overline{x}_3, x_4), (\overline{x}_3, \overline{x}_4)\}.$$

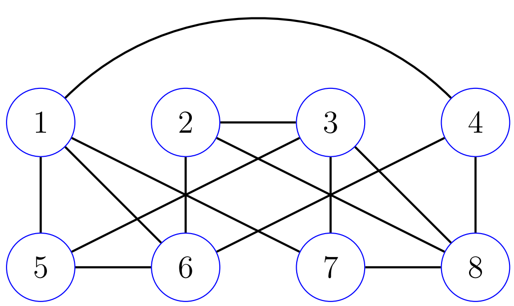
(b) Apply the improved `2SAT` algorithm to obtain a satisfying assignment for $\mathcal{C}$. When deciding on the next reachability set R_l to compute, follow the literal order $l = x_1, \overline{x}_1, \ldots, x_4, \overline{x}_4$. For each consistent reachability set encountered, provide the partial assignment α_{R_l} associated with R_l and draw the reduced implication graph before continuing to the next reachability set. Note: do *not* compute the reachability set for a literal that has already been assigned a truth value. Provide a final assignment α and verify that it satisfies all six clauses.

(c) Suppose a `Reachability`-oracle answers "yes" to the query reachable$(G_{\mathcal{C}}, \overline{x}_2, x_2)$. If $\mathcal{C}$ is satisfiable via assignment α, then what is the value of $\alpha(x_2)$? Explain.

LO9. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction fromm problem A to problem B.

(b) Suppose $(G, k = 3)$ is an instance of the `Clique` decision problem, where G is drawn below. Draw $f(G, k)$, where f is the mapping reduction from `Clique` to the `Half Clique` decision problem.



(c) Verify that f is valid for input (G, k) in the sense that both (G, k) and $f(G)$ are either both positive instances or both negative instances of their respective problems. Defend your answer.