CECS 528, Solutions to Learning Outcome Assessment 9 Problems, Pink, Fall 2023, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems

LO5. Answer the following with regards to a correctness-proof outline for Prim's algorithm.

(a) In the correctness proof of Prim's algorithm, suppose $T = e_1, \ldots, e_{n-1}$ are the edges selected by Prim's algorithm (in that order) and T_{opt} is an mst that uses edges $e_1, \ldots, e_{k-1}$, but for which $e_k \notin T_{\text{opt}}$. Explain why e_k is incident with one vertex in T_{k-1} (Prim's tree after round $k-1$) and with one vertex *not* in T_{k-1}. Hint: your answer should have *nothing* to do with the fact that $e_k \notin T_{\text{opt}}$.

Solution. Since e_k is added to T_{k-1} in Round k of Prim's algorithm, it must be incident with exactly one vertex in T_{k-1}. Otherwise, it would not be a candidate for selection in Round k.

(b) Since $e_k \notin T_{\text{opt}}$, it follows that $T_{\text{opt}} + e_k$ has a cycle C. Explain why there must be an edge $e \in C$ for which i) $e \neq e_k$ and ii) e is incident with one vertex in T_{k-1} and with one vertex *not* in T_{k-1}. Furthermore, explain why $w(e) \geq w(e_k)$.

Solution. Since cycle C possesses an edge e_k that is incident with some vertex not in T_{k-1} and also some vertex that is in T_{k-1}, it follows that the cycle enters T_{k-1} via e_k and thus must exit T_{k-1} via some other edge, call it e. Hence, e is connected to T_{k-1} and is thus a candidate for selection during Round k. However, since e_k was chosen over e, it follows that $w(e_k) \leq w(e)$.

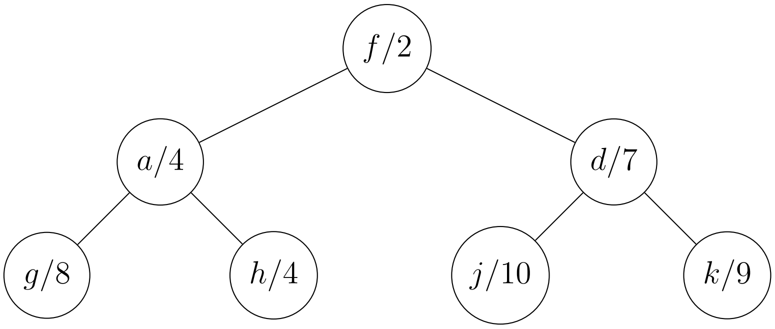
(c) Explain why $T_{\text{opt}} - e + e_k$ is also an mst, i.e. a tree of minimum cost.

Solution. Since e and e_k belong to the only cycle of connected graph $T_{\text{opt}} + e_k$, removing e keeps the graph connnected but now the graph becomes acyclic, hence a tree. Also, it is an mst since $w(e_k) \leq w(e)$, so its cost does not exceed (in fact equals) that of T_{opt}.

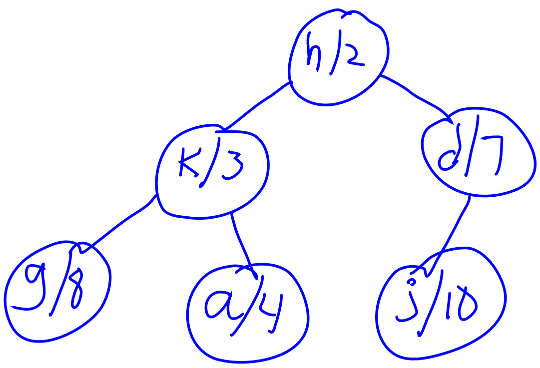
LO6. The tree below shows the state of the binary min-heap at the beginning of some round of Prim's algorithm, applied to some weighted graph G. If G has edges

$$(a, f, 6), (d, g, 4), (f, g, 10), (f, h, 2), (f, k, 3),$$

then draw a plausible state of the heap at the end of the round.



Solution.



LO7. Answer the following.

(a) Provide the dynamic-programming recurrence for computing the distance $\mathrm{D}(u, v)$, from a vertex u to a vertex v in a directed acyclic graph (DAG) $G = (V, E, c)$, where $c(x, y)$ gives the cost of edge $e = (x, y)$, for each $e \in E$. The recurrence should allow one to compute the distance from a single source to all other vertices in a linear number of steps. Hint: step backward from v.

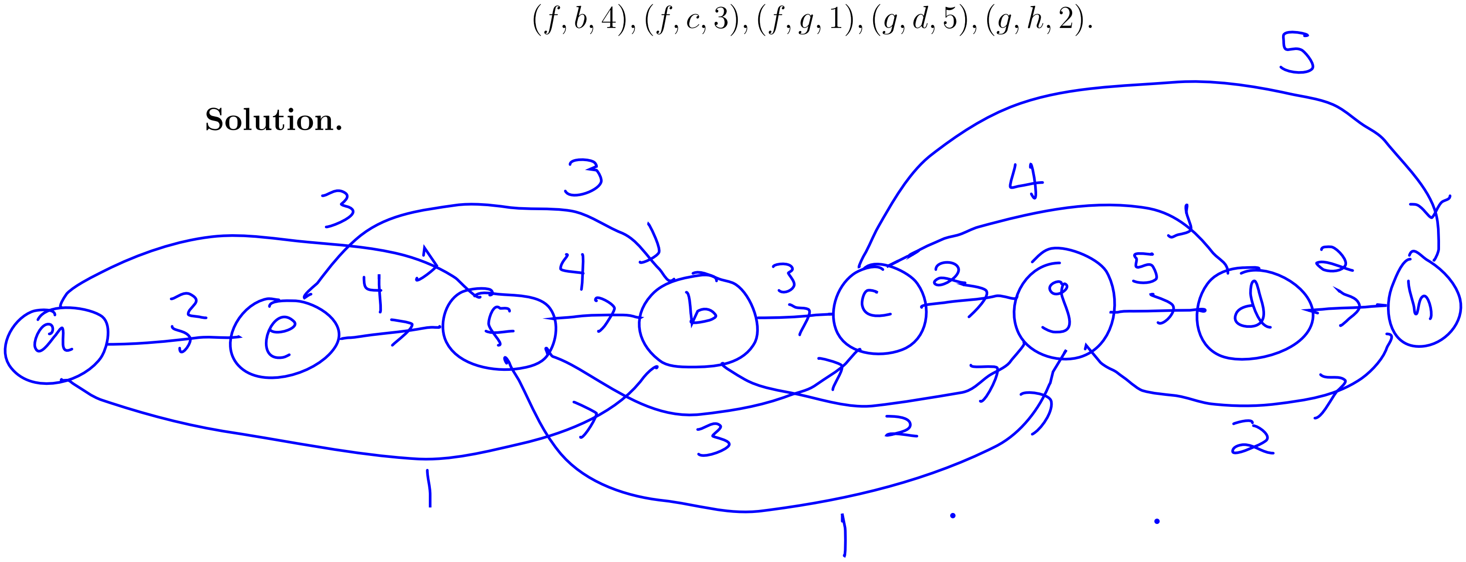
Solution.

$$\mathrm{D}(u,v) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } \deg^{+}(v) = 0 \\ \min_{(w,v)\in E}(c(w,v) + \mathrm{D}(u,w)) & \text{otherwise} \end{cases}$$

(b) Draw the vertices of the following DAG G in a linear left-to-right manner so that the vertices are topologically sorted, meaning, if (u, v) is an edge of G, then u appears to the left of v. The vertices of G are a-h, while the weighted edges of G are

$$(a, b, 1), (a, e, 2), (a, f, 3), (b, c, 3), (b, g, 2), (c, d, 4), (c, g, 2), (c, h, 5), (d, h, 2), (e, b, 3), (e, f, 4),$$
$$(f, b, 4), (f, c, 3), (f, g, 1), (g, d, 5), (g, h, 2).$$

Solution.



(c) Starting from left to right in topological order, use the recurrence to compute

$$d(a, a), \ldots, d(a, h).$$

Solution.

$$D(a, a) = 0.$$

$$D(a, e) = 2.$$

$$D(a, f) = \min(4 + D(a, e), 3 + D(a, a)) = \min(4 + 2, 3 + 0) = 3.$$

$$D(a, b) = \min(1 + D(a, a), 3 + D(a, e), 4 + D(a, f)) = \min(1 + 0, 3 + 2, 4 + 3) = 1.$$

$$D(a, c) = \min(3 + D(a, b), 3 + D(a, f)) = \min(3 + 1, 3 + 3) = 4.$$

$$D(a, g) = \min(2 + D(a, b), 2 + D(a, c), 1 + D(a, f)) = \min(2 + 1, 2 + 4, 1 + 3) = 3.$$

$$D(a, d) = \min(5 + D(a, g), 4 + D(a, c)) = \min(5 + 3, 4 + 4) = 8.$$

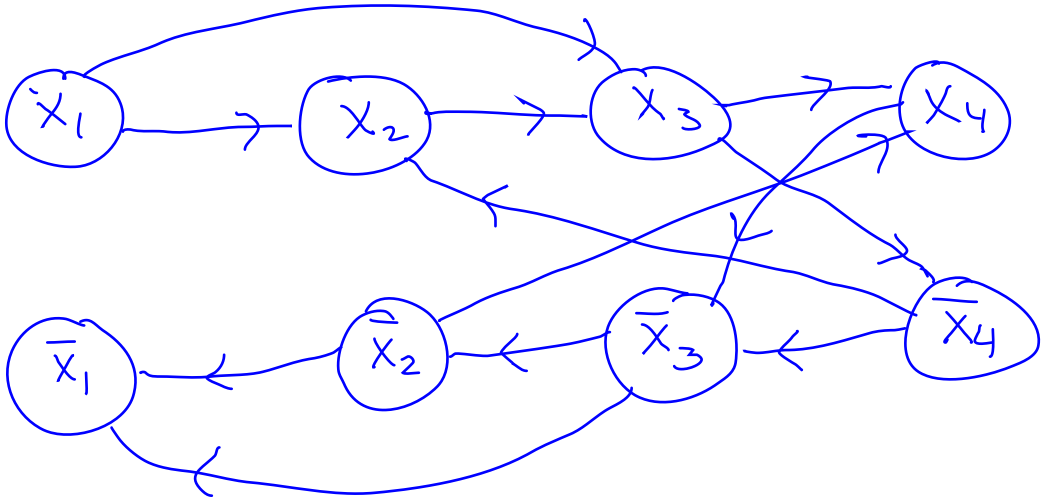
$$D(a, h) = \min(5 + D(a, c), 2 + D(a, d), 2 + D(a, g)) = \min(5 + 4, 2 + 8, 2 + 3) = 5.$$

LO8. Do/answer the following.

(a) Draw the implication graph $G_{\mathcal{C}}$ associated with the `2SAT` instance

$$\mathcal{C} = \{(\overline{x}_1, x_2), (\overline{x}_1, x_3), (\overline{x}_2, x_3), (x_2, x_4), (\overline{x}_3, x_4), (\overline{x}_3, \overline{x}_4)\}.$$

Solution.



(b) Apply the improved `2SAT` algorithm to obtain a satisfying assignment for $\mathcal{C}$. When deciding on the next reachability set R_l to compute, follow the literal order $l = x_1, \overline{x}_1, \ldots, x_4, \overline{x}_4$. For each consistent reachability set encountered, provide the partial assignment α_{R_l} associated with R_l and draw the reduced implication graph before continuing to the next reachability set. Note: do *not* compute the reachability set for a literal that has already been assigned a truth value. Provide a final assignment α and verify that it satisfies all six clauses.

Solution. $R_{x_1} = \{x_1, \overline{x}_1, \ldots, x_4, \overline{x}_4\}$ is inconsistent, while $R_{\overline{x}_1} = \{\overline{x}_1\}$ which is consistent and so $\alpha_{R_{\overline{x}_1}} = (x_1 = 0)$. Now remove x_1 and $\overline{x}_1$ from the implication graph including all edges incident with these vertices. Then $R_{x_2} = \{x_2, \overline{x}_2, \ldots, x_4, \overline{x}_4\}$ is inconsistent, while $R_{\overline{x}_2} = \{\overline{x}_2, \overline{x}_3, x_4\}$ which is consistent and so $\alpha_{R_{\overline{x}_2}} = (x_2 = 0, x_3 = 0, x_4 = 1)$. Final assignment:

$$\alpha = \alpha_{R_{\overline{x}_1}} \cup \alpha_{R_{\overline{x}_2}} = (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1).$$

(c) Suppose a `Reachability`-oracle answers "yes" to the query reachable$(G_{\mathcal{C}}, \overline{x}_2, x_2)$. If $\mathcal{C}$ is satisfiable via assignment α, then what is the value of $\alpha(x_2)$? Explain.

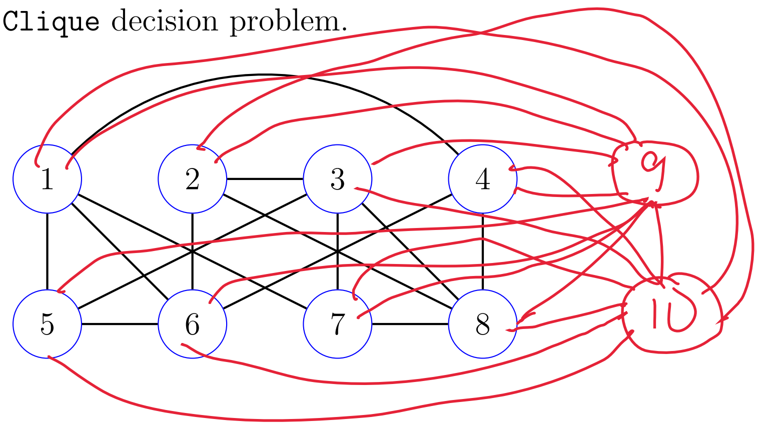
Solution. $\alpha(x_2) = 1$ since $R_{\overline{x}_2}$ is inconsistent (it contains x_2) and, since $\mathcal{C}$ is satisfiable, it must be the case that R_{x_2} is consistent, in which case any satisfying assignment α must satisfy $\alpha(x_2) = 1$.

LO9. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction fromm problem A to problem B.

Solution. See Definition 2.1 of the Mapping Reducibility lecture.

(b) Suppose $(G, k = 3)$ is an instance of the `Clique` decision problem, where G is drawn (in black) below. Draw $f(G, k)$, where f is the mapping reduction from `Clique` to the `Half Clique` decision problem.



Solution. $G' = f(G, k)$ is drawn above as G with two additonal (red) vertices and ~~nine~~ additional (red) edges.

(c) Verify that f is valid for input (G, k) in the sense that both (G, k) and $f(G, k)$ are either both positive instances or both negative instances of their respective problems. Defend your answer.

Solution. Both (G, k) and $G' = f(G, k)$ are positive instances since, e.g., $C = \{1, 5, 6\}$ is a 3-clique for G while $C' = \{1, 5, 6, 9, 10\}$ is a half-clique for G'.