CECS 528, Learning Outcome Assessment 6, Pink, Fall 2023, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems

LO2. Solve the following problems.

(a) Use the Master Theorem to determine the growth of $T(n)$ if it satisfies the recurrence $T(n) = 4T(n/2) + n^{\log_4 8}$. Defend your answer.

(b) Use the substitution method to prove that, if $T(n)$ satisfies

$$T(n) = 4T(n/2) + 4n^{2.5}$$

then $T(n) = \mathrm{O}(n^{2.5})$.

LO3. Solve the following problems.

(a) For the randomized version of the `Find-Statistic` algorithm, explain the rationale behind the recurrence

$$T(n) \leq T(3n/4) + \mathrm{O}(n)$$

for $T(n)$. What does $T(n)$ represent in this recurrence?

(b) Consider the following algorithm called `multiply` for multiplying two n-bit binary numbers x and y, where we assume n is even. Let x_L and x_R be the leftmost $n/2$ and rightmost $n/2$ bits of x respectively. Define y_L and y_R similarly. Let P_1 be the result of calling `multiply` on inputs x_L and y_L, P_2 be the result of calling `multiply` on inputs x_R and y_R, and P_3 the result of calling `multiply` on inputs $x_L + x_R$ and $y_L + y_R$. Then return the value $P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$. For the two binary integers $x = 10100101$ and $y = 11110000$, determine the values of P_1, P_2, and P_3 at the root level of recursion, and verify that $xy = P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$. Hint: you may evaluate P_1, P_2, and P_3 non-recursively using base-10.

LO4. Answer the following.

(a) Recall that the nth roots of unity (n a positive power of 2) come in additive-inverse pairs. Why is this fact essential in order for the FFT algorithm to work in a log-linear number of steps.

(b) Compute $\mathrm{DFT}_4^{-1}(5, 2, 3, -4)$ using the IFFT algorithm. Show the solution to each of the seven subproblem instances and, for each one, clearly represent it using DFT^{-1} notation and apply the formula for computing it. Show all work.

LO5. Answer the following with regards to a correctness-proof outline for Dijkstra's algorithm.

(a) In relation to Dijkstra's algorithm, provide a definition for what it means to be i) an i-neighboring path from source s to an external vertex v, and ii) the i-neighboring distance $d_i(s, v)$ from source s to external vertex v. Hint: at this point in the algorithm i nodes have been added to the DDT.

(b) Using the definitions from part a, describe the greedy choice that is made in each round of Dijkstra's algorithm.

(c) If vertex v is chosen by Dijkstra in Round $i+1$, use part b to prove that $d(s, v) = d_i(s, v)$. Hint: if i-neighboring path P from s to v has cost $d_i(s, v)$ and Q is any other path from s to v, explain why $\text{cost}(Q) \geq d_i(s, v)$.

LO6. For the weighted graph with edges

$$(a, e, 4), (b, e, 1), (c, e, 5), (c, f, 2), (d, e, 6), (e, f, 3),$$

Show how the forest of union-find trees changes when processing each edge in Kruskal's sorted list of edges. When unioning two trees, use the convention that the root of the union is the root which has the *lower* alphabetical order. For example, if two trees, one with root a, the other with root b, are to be unioned, then the unioned tree should have root a.