CECS 528, Learning Outcome 12 Assessment, December 6th 2023, Dr. Ebert

Directions: Turn in handwritten solutions on the morning of the December 18th final exam. Order your solutions by problem number/letter, and do *not* write on the back of any page. Only the front side of each page will be read. Staple all pages (no folding of corners). Although it's OK to discuss problems with other students, plagiarism will not be tolerated and will result in a final course grade of F. Make sure to describe solutions in your own words. All problems are equally weighted and must be passed in order to pass LO12.

Problems

1. Consider the wheel graph W_n which has vertex set $V = \{c, p_1, p_2, \ldots, p_{2n-1}, p_{2n}\}$, and edge set $E = E_{\text{perim}} \cup E_{\text{spoke}}$ where $E_{\text{perim}} = \{(p_1, p_2), (p_2, p_3), \ldots, (p_{2n-1}, p_{2n}), (p_{2n}, p_1)\}$ and $E_{\text{spoke}} = \{(c, p_1), (c, p_3), \ldots, (c, p_{2n-1})\}$.

   (a) For the `Vertex Cover` approximation algorithm presented in lecture, determine the resulting cover and approximation ratio when the algorithm is applied to W_n and the edges are presented to the algorithm as

   $$(c, p_1), (c, p_3), \ldots, (c, p_{2n-1}), (p_1, p_2), (p_2, p_3), \ldots, (p_{2n-1}, p_{2n}).$$

   Explain your reasoning.

   (b) Repeat the previous problem but now assume the algorithm being used is the greedy algorithm described on Page 4 of the Approximation-Algorithms lecture. Explain your reasoning.

2. Apply the `Clustering` approximation algorithm to the points

   $$A = (0, 0), B = (6, 0), C = (7, 2), D = (3, 6), E = (4, 5), F = (0, 9), G = (9, 9), H = (7, 8), I = (7, 6),$$

   $$J = (3, 8), K = (2, 4), L = (7, 4),$$

   with $k = 4$. Assume the metric used is the **taxi-cab metric**, where

   $$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

   Choose $A = (0, 0)$ to be the first cluster center.

   (a) Plot the points on a 2-dimensional grid.

   (b) For each of the three iterations of the algorithm loop, provide a table showing $d(x, P)$ for each candidate center x. Highlight the candidate that is chosen as a cluster center.

   (c) Provide the four cluster sets and the maximum diameter of any of the sets.

(d) Provide a lower bound r for the maximum diameter of a cluster of an optimal 4-cluster solution. Explain how you derived this bound.

(e) Provide a 4-cluster that you believe is likely optimal and provide its maximum diameter. Compare the maximum diameter of the 4-cluster returned by the Clustering algorithm with the maximum diameter of your "optimal cluster".

3. Use the points and metric from Problem 2 to demonstrate the two approximation algorithms, `MST` and `Christofides`, for TSP. Assume that the Hamilton Cycles are to begin and end at point A.

   (a) Plot the points on a 2-dimensional grid and highlight an mst T for the TSP graph G whose vertices are $\{A, \ldots, L\}$. Determine the cost of this mst.

   (b) Provide the path P that is traversed during a depth-first traversal of T that starts and ends at vertex A. Hint: every edge of T should appear exactly twice in P. Use P to compute a HC-tour C of the vertices of G. Provide C's cost and use it to determine a cost interval that contains the cost of the optimal HC-tour.

   (c) Redraw the 2-dimensional grid with points and mst, but now also includes the min-cost matching M of edges that provide a pairing of the odd-degree vertices of T. Highlight the matching edges using a different color than what is used for the mst edges.

   (d) Provide an Euler circuit E that begins and ends at A and visits every edge of $T \cup M$ exactly once. Hint: any edge that is in $T \cap M$, will need to be traversed twice. Use E to compute a HC-tour C of the vertices of G. Provide C's cost and use it to determine a cost interval that contains the cost of the optimal HC-tour.

   (e) Provide a HC-tour of the vertices of G that you believe to be optimal. Provide its cost and compare it with the cost of the HC-tour that was determined using Christofides' algorithm.