CECS 528, Learning Outcome Assessment Makeup Problems, Yellow, December 6th, 2023, Dr. Ebert

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION ALLOWED. Submit each solution on a separate sheet of paper.

Problems

LO1. Complete the following problems.

(a) Demonstrate each step (line) of Euclid's algorithm on inputs $a = 63$ and $b = 28$. Then work backwards to provide a linear combination of 54 and 14 that sums to their gcd.

(b) For the Strassen-Solovay primality test with $n = 14$, determine whether or not $a = 2$ is a witness to n not being prime. Do this by evaluating *both* sides of the test congruence, mod 14.

LO2. Solve the following.

(a) Use the Master Theorem to determine the growth of $T(n)$ if it satisfies the recurrence $T(n) = 4T(n/2) + n^{\log_4 16}$. Defend your answer.

(b) Use the substitution method to prove that if $T(n)$ satisfies

$$T(n) = 4T(n/2) + 6n$$

then $T(n) = \Omega(n^2)$.

LO3. Answer/solve the following.

(a) For the randomized version of the `Find-Statistic` algorithm, explain the rationale behind the recurrence

$$T(n) \leq T(3n/4) + \mathrm{O}(n)$$

for $T(n)$. What does $T(n)$ represent in this recurrence?

(b) Consider the following algorithm called `multiply` for multiplying two n-bit binary numbers x and y, where we assume n is even. Let x_L and x_R be the leftmost $n/2$ and rightmost $n/2$ bits of x respectively. Define y_L and y_R similarly. Let P_1 be the result of calling `multiply` on inputs x_L and y_L, P_2 be the result of calling `multiply` on inputs x_R and y_R, and P_3 the result of calling `multiply` on inputs $x_L + x_R$ and $y_L + y_R$. Then return the value $P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$. For the two binary integers $x = 10100101$ and $y = 11110000$, determine the values of P_1, P_2, and P_3 at the root level of recursion, and verify that $xy = P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$. Hint: you may evaluate P_1, P_2, and P_3 non-recursively using base-10.

LO4. Answer the following.

(a) The FFT algorithm owes its existence to what two properties that are possessed by the nth roots of unity when n is even?

(b) Compute $\text{DFT}_4^{-1}(6, -3, 1, -8)$ using the IFFT algorithm. Show the solution to each of the seven subproblem instances and, for each one, clearly represent it using DFT^{-1} notation and apply the formula for computing it. Show all work.

LO5. Answer the following with regards to a correctness-proof outline for Prim's algorithm.

(a) In the correctness proof of Prim's algorithm, suppose $T = e_1, \ldots, e_{n-1}$ are the edges selected by Prim's algorithm (in that order) and T_{opt} is an mst that uses edges $e_1, \ldots, e_{k-1}$, but for which $e_k \notin T_{\text{opt}}$. Explain why e_k is incident with one vertex in T_{k-1} (Prim's tree after round $k-1$) and with one vertex *not* in T_{k-1}. Hint: your answer should have *nothing* to do with the fact that $e_k \notin T_{\text{opt}}$.

(b) Since $e_k \notin T_{\text{opt}}$, it follows that $T_{\text{opt}} + e_k$ has a cycle C. Explain why there must be an edge $e \in C$ for which i) $e \neq e_k$ and ii) e is incident with one vertex in T_{k-1} and with one vertex *not* in T_{k-1}. Furthermore, explain why $w(e) \geq w(e_k)$.

(c) Explain why $T_{\text{opt}} - e + e_k$ is also an mst, i.e. a tree of minimum cost.

LO6. For the weighted graph with edges

$$(a, e, 6), (b, e, 4), (c, e, 3), (c, d, 5), (d, f, 2), (e, f, 1),$$

Show how the disjoint-set data structure forest changes when processing each edge in Kruskal's sorted list of edges. When unioning two trees, use the convention that the root of the union is the root which has the *lower* alphabetical order. For example, if two trees, one with root a, the other with root b, are to be unioned, then the unioned tree should have root a.

LO7. Solve the following problems.

(a) The dynamic-programming algorithm that solves the `Longest Common Subsequence (LCS)` optimization problem defines a recurrence for the function $\text{lcs}(i, j)$. In words, what does $\text{lcs}(i, j)$ equal? Hint: do *not* write the recurrence (see Part b).

(b) Provide the dynamic-programming recurrence for $\text{lcs}(i, j)$.

(c) Apply the recurrence from Part b to the words $u = \text{baabab}$ and $v = \text{bbbaaa}$. Show the matrix of subproblem solutions and use it to provide an optimal solution.

LO8. Do/answer the following.

(a) Draw the implication graph $G_{\mathcal{C}}$ associated with the `2SAT` instance

$$\mathcal{C} = \{(x_1, x_2), (\overline{x}_1, \overline{x}_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (\overline{x}_2, \overline{x}_3), (x_2, x_4), (\overline{x}_3, x_5)\}.$$

(b) Apply the improved `2SAT` algorithm to obtain a satisfying assignment for $\mathcal{C}$. When deciding on the next reachability set R_l to compute, follow the literal order $l = x_1, \overline{x}_1, \ldots, x_5, \overline{x}_5$. For each consistent reachability set encountered, provide the partial assignment α_{R_l} associated with R_l and draw the reduced implication graph before continuing to the next reachability set. Note: do *not* compute the reachability set for a literal that has already been assigned a truth value. Provide a final assignment α and verify that it satisfies all the clauses.

(c) Given an instance $\mathcal{C}$ of `2SAT` having m clauses and n variables, when executing the original `2SAT` algorithm, what is the worst-case number of queries to a `Reachability` oracle that is needed before one can conclude that $\mathcal{C}$ is unsatisfiable? Explain.

LO9. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from decision problem A to decision problem B.

(b) In relation to your answer to part a, if $f(n)$ is a valid mapping reduction from the `Even` decision problem to the `Odd` decision problem, then, if n is even, then what must be true about $f(n)$? Explain.

(c) Is $f(n) = n^2 + 5n + 9$ a valid mapping reduction from the `Even` decision problem to the `Odd` decision problem? Justify your answer.

LO10. An instance of `Set Cover (SC)` is a triple $(\mathcal{S}, m, k)$, where $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a collection of n subsets, where $S_i \subseteq \{1, \ldots, m\}$, for each $i = 1, \ldots, n$, and a nonnegative integer k. The problem is to decide if there are k subsets $S_{i_1}, \ldots, S_{i_k}$ for which

$$S_{i_1} \cup \cdots \cup S_{i_k} = \{1, \ldots, m\}.$$

(a) For a given instance $(\mathcal{S}, m, k)$ of `SC` describe a certificate in relation to $(\mathcal{S}, m, k)$.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance $(\mathcal{S}, m, k)$, and ii) a certificate for $(\mathcal{S}, m, k)$ as defined in part a, and decides if the certificate is valid for $(\mathcal{S}, m, k)$.

(c) Provide appropriate size parameters for `SC`. Hint: there are two of them.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer.

LO11. Answer the following.

(a) Recall the mapping reduction $f(\mathcal{C}) = (S, t)$, where f maps an instance of `3SAT` to an instance of the `Subset Sum` decision problem. Given `3SAT` instance

$$\mathcal{C} = \{c_1 = (x_1, x_2, x_4), c_2 = (\overline{x}_2, x_3, \overline{x}_4), c_3 = (\overline{x}_1, x_2, x_3), c_4 = (\overline{x}_1, \overline{x}_3, \overline{x}_4),$$

What is the cardinality of S? What is the value of t? Explain and/or Show work. (5 pts)

(b) Continuing from part a, is (S, t) a positive intance? If not, explain why. If yes, provide a subset of S that sums to target t. (6 pts)

(c) Recall the mapping reduction $f(G) = (G', k)$, where f maps an instance of `Hamilton Cycle` to an instance of the `TSP` decision problem. If G has 8 vertices, then how many edges does G' have? What is the value of k? Explain. (7 pts)

(d) Continuing from part c, if G has a Hamilton path, but no Hamilton cycle, then what is the total cost of the least-cost Hamilton cycle in G'. In other words, determine the cost of the most cost-efficient cycle that appears in G'. Explain. (7 pts)