CECS 329, Exam 1, September 21st, Fall 2023, Dr. Ebert

Rules for Completing the Problems

NO NOTES, BOOKS, ELECTRONIC DEVICES, OR INTERPERSONAL COMMUNICATION allowed when solving these problems. Make sure all these items are put away BEFORE looking at the problems. FAILURE TO ABIDE BY THESE RULES MAY RESULT IN A FINAL COURSE GRADE OF F.

Directions

Choose up to **six problems** to solve. Clearly mark each problem you want graded by placing an 'X' or check mark in the appropriate box. **If you don't mark any problems for us to grade or mark 7 or more problems, then we will record grades for the 6 that received the *fewest* points.**

Problem	1	2	3	4	5	6
Grade?						
Result						

Your Full Name:

Your Class ID:

1. Answer the following. Note: correctly solving this problem counts for passing LO1.

   a. Provide the definition of what it means to be a mapping reduction from decision problem A to decision problem B. (5 pts)

      Solution. See Definition 2.1 of Map Reducibility lecture.

   b. For the mapping reduction f : `Subset Sum` $\rightarrow$ `Set Partition`, based on the reduction provided in lecture, determine $f(S, t)$ for `Subset Sum` instance

      $$(S = \{4, 7, 15, 19, 22, 38, 44, 45\}, t = 111).$$

      Show work. (10 pts)

      Solution. We have

      $$f(S, t) = S' = S \cup \{J\},$$

      where $J = 2t - M = 222 - 194 = 28$.

   c. Verify that $A = \{22, 44, 45\}$ is a valid certificate for (S, t), and use A to construct a valid certificate for $f(S, t)$. Explain and show work. (10 pts)

      Solution. The members of A sum to $t = 111$, and so A is a valid certificate for (S, t). Also, A and $B = \{4, 7, 15, 19, J = 28, 38\}$ partition S' and both sets sum to 111. Therefore, (A, B) is a valid certificate for S'.

2. An instance of `Set Cover` is a triple $(\mathcal{S}, m, k)$, where $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a collection of n subsets, where $S_i \subseteq \{1, 2, \ldots, m\}$, for each $i = 1, \ldots, n$, and k is a nonnegative integer. The problem is to decide if there are k subsets $S_{i_1}, \ldots, S_{i_k}$ for which

$$S_{i_1} \cup \cdots \cup S_{i_k} = \{1, 2, \ldots, m\}.$$

Note: correctly solving this problem counts for passing LO2.

a. Verify that $(\mathcal{S}, m, k)$ is a positive instance of `Set Cover`, where $m = 9$, $k = 4$, and

$$\mathcal{S} = \{\{1,3,5\}, \{3,7,9\}, \{2,4,5\}, \{2,6,7\}, \{6,7,9\}, \{2,7,9\}, \{1,3,7\}, \{4,5,8\}\}.$$

Solution. $S_5 \cup S_6 \cup S_7 \cup S_8 = \{1, \ldots, 9\}$ and so we have a positive instance.

b. For a given instance $(\mathcal{S}, m, k)$ of `Set Cover` describe a certificate in relation to $(\mathcal{S}, m, k)$. (5 pts)

Solution. $C \subseteq \mathcal{S}$ is a subset of size k.

c. Provide a semi-formal verifier algorithm that takes as input i) an instance $(\mathcal{S}, m, k)$, ii) a certificate for G as defined in part b, and decides if the certificate is valid. (5 pts)

Solution.

Initialize array $a[1:m]$ so that $a[i] = 0$ for all $i = 1, \ldots, m$.
For each $S \in C$,
    For each $s \in S$,
        $a[s] = 1$.
Return $\prod_{i=1}^{m} a[i]$.

d. Provide size parameters that may be used the measure the size of an instance of `Set Cover`. Hint: there should be two of them. (5 pts)

Solution. $n = |\mathcal{S}|$, and m which is a bound on the size of each subset of $\mathcal{S}$.

e. Use the size parameters from part d to describe the running time of your verifier from part c. Defend your answer in relation to the algorithm you provided for the verifier. (5 pts)

Solution. The outer loop requires $O(n)$ iterations while the inner loop requires $O(m)$ iterations (since each subset of $\mathcal{S}$ has at most m members), for a total of $O(mn)$ steps since there is only a single assignment within the nested loop (an array assignment that requires $O(1)$ steps). The final `return` statement requires $O(m)$ steps for a total of $O(mn)$ steps which represents quadratic growth. Therefore, `Set Cover` $\in$ `NP`.

3. Answer the following. Note: correctly solving at least three parts counts for passing LO3.

   a. Recall the mapping reduction $f(\mathcal{C}) = (S, t)$, where f maps an instance of `3SAT` to an instance of the `Subset Sum` decision problem. Given `3SAT` instance

   $$\mathcal{C} = \{c_1 = (x_1, x_2, x_4), c_2 = (\overline{x}_2, x_3, \overline{x}_4), c_3 = (\overline{x}_1, x_2, x_3), c_4 = (\overline{x}_1, \overline{x}_3, \overline{x}_4),$$

   What is the cardinality of S? What is the value of t? Explain and/or Show work. (5 pts)

   Solution. We have

   $$f(\mathcal{C}) = (S, t),$$

   where $|S| = 2m + 2n = 2(4) + 2(4) = 16$.

   b. Continuing from part a, is (S, t) a positive intance? If not, explain why. If yes, provide a subset of S that sums to target t. (6 pts)

   Solution. Yes, since $\alpha = (x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1)$ satisfies $\mathcal{C}$ and a positive instance of `3SAT` must map to a positive instance of `SS`. Moreover, the assignment tells us that

   $$A = \{z_1, y_2, y_3, y_4, g_1, g_2, h_2, g_4, h_4\}$$

   sums to $t = 11113333$. Note: set A will vary depending on the chosen satisfying assignment.

   c. Recall the mapping reduction $f(G) = (G', k)$, where f maps an instance of `Hamilton Cycle` to an instance of the `TSP` decision problem. If G has 8 vertices, then how many edges does G' have? What is the value of k? Explain. (7 pts)

   Solution. G' is a complete graph, and thus will have $8(7)/2 = 28$ edges.

   d. Continuing from part c, if G has a Hamilton path, but no Hamilton cycle, then what is the total cost of the least-cost Hamilton cycle in G'. In other words, determine the cost of the most cost-efficient cycle that appears in G'. Explain. (7 pts)

   Solution. The least-cost cycle in G' would use the length-7 Hamilton path for a total path cost of 7, plus one additional edge with cost 8, for a total cycle cost of 15.

4. Answer the following. Note: scoring 14 or more points counts for passing LO4.

   a. Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

      i. An instance of `Triangle` is a simple graph G and the problem is to decide if G has a 3-clique. **Solution.** P

      ii. An instance of `Tired Traveling Salesperson (TTSP)` is a complete weighted graph G and an integer $k \geq 0$. The problem is to decide if every Hamilton Cycle in G has a cost that exceeds k. **Solution.** co-NP

      iii. An instance of `Composite` is a natural number $n \geq 2$, and the problem is to decide if n is a composite number, i.e. is divisible by a number $2 \leq d \leq n-1$. **Solution.** P

      iv. An instance of `Quadratic Diophantine` is a triple of positive integers (a, b, c) and the problem is to decide if there are positive integers x and y for which $ax^2 + by = c$. **Solution.** NP, but P is worth 1 point.

   b. Circle the correct answer. Which of the following certificates is best suited for checking if some simple graph $G = (V, E)$ is a positive instance of `Hamilton Cycle`. (8 points)

      i. Certificate C is a permutation of all n vertices of G.

      ii. Certificate C is a subset of n edges of G, where $n = |V|$.

      iii. Certificate C is a permutation of all m edges of G.

      iv. Certificate C is a permutation of n edges of G, where $n = |V|$.

      Solution. Either i or iv is acceptable.

   c. Circle the correct answer. Which of the following is a true statement (9 points)

      i. If $A \leq_m^p B$, $A \in \mathtt{P}$, and B is NP-complete, then $B \in \mathtt{P}$.

      ii. If $A \leq_m^p B$, A is NP-complete, $B \in \mathtt{NP}$, then $\mathtt{P} \neq \mathtt{NP}$.

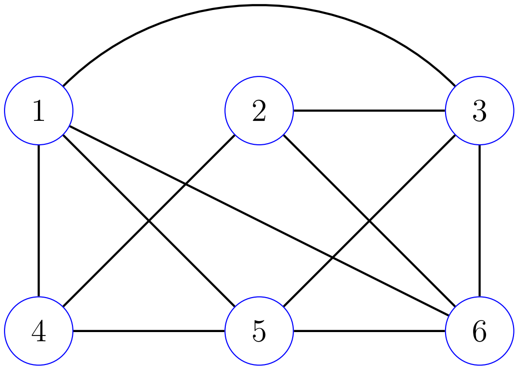
      iii. If $A \leq_m^p B$, A is NP-complete, $B \in \mathtt{P}$, then $\mathtt{P} = \mathtt{NP}$.

      iv. If $A \leq_m^p B$, $A \in \mathtt{P}$, and B is NP-complete, then $\mathtt{P} = \mathtt{NP}$.

      Solution. iii, since any NP problem A can be solved in polynomial time by first applying the polynomial-time map reduction from an instance a of A to an instance b of B, and then using the polynomial-time algorithm for B to solve instance b. Finally, because a map reduction was applied to a, the answer computed for b will equal the answer to a.

5. In lecture we showed a polynomial-time mapping reduction $f : \texttt{HP} \rightarrow \texttt{HC}$.

   a. Draw the image of $f(G, 1, 6)$, where G is the graph shown below. Hint: you may represent your image by modifying G. (5 pts)



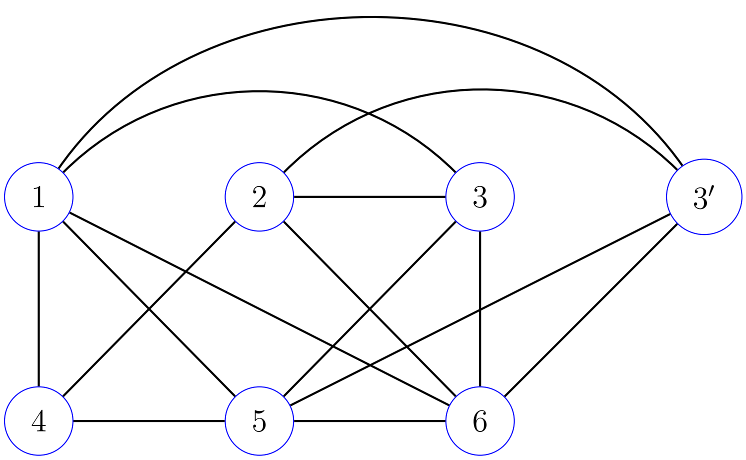
   Solution. $f(G, a, b) = G'$ where G' is the graph G but with a new vertex v that is made adjacent to both a and b.

   b. Provide a complete description of a valid polynomial-time mapping reduction $g : \texttt{HC} \rightarrow \texttt{HP}$ from `Hamilton Cycle` to `Hamilton Path`. Function g must map a graph G to a triple (G', a, b), where a and b are vertices of G' and G has a Hamilton Cycle iff there is a Hamilton path from a to b. Use the graph from part a to demonstrate your reduction. Hint: clone a vertex of G. Argue that your reduction is valid. (20 pts)

   Solution. Let v be an arbitrary vertex of G. Then $g(G) = (G', v, v')$, where G' is the graph G, but with an added vertex v' which is the clone of v in the sense that, for any edge (v, w) of G, there is also an edge (v', w) in G'.

   First suppose that G has a Hamilton Cycle. Then there is a cycle that starts and ends at v and visits every other vertex exactly once. Then G' has a Hamilton path from v to v' by using the aforementioned cycle, but the path at v', rather than returning to v. This is possible since the second-to-last vertex u visited in the cycle, is adjacent to v' since (u, v) is an edge of G. Conversely, if G' has a Hamilton path from v to v'. Then G has a Hamilton cycle, since the second-to-last vertex u in the Hamilton path must be adjaccent to v since it is adjacent to v'.

   The figure below shows $f(G)$ where G is the graph from part a, and $v = 3$ is the cloned vertex.

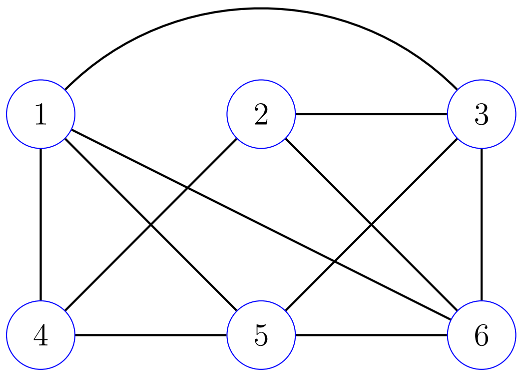


6. The `Half Clique` decision problem is the problem of deciding if a simple graph $G = (V, E)$ has a Clique of size $|V|/2$.

   a. Describe an embedding reduction f from `Half Clique` to `Clique`. (10 points)

   Solution. $f(G) = (G, k = |V|/2)$, where V is G's vertex set.

   b. Now describe a contraction reduction from `Clique` to `Half Clique` and demonstrate it for the `Clique` instance $(G, k = 4)$, where G is the graph shown below. (15 points)



   Solution. See solution to Exercise 13 from the Map Reducibility lecture. For the above graph, since $k = 4 > 6/2$, $f(G)$ is obtained by adding $2k - |V| = 8 - 6 = 2$ isolated vertices to G. This will ensure that G has a 4-clique iff $f(G)$ has a half-clique of size $8/2 = 4$.