CECS 329, Learning Outcome Assessment 6, October 12th, Fall 2023, Dr. Ebert

Problems

LO2. An instance of `Dominating Set` is a simple graph $G = (V, E)$ and a nonnegative integer $k \geq 0$, and the problem is to decide if there is a set $D \subseteq V$ of k vertices in G for which, for every vertex $u \in V - D$, there is vertex $v \in D$ for which $(u, v) \in E$. In other words, every vertex not in D is adjacent to some vertex in D.

(a) For a given instance (G, k) of `Dominating Set` describe a certificate in relation to (G, k).

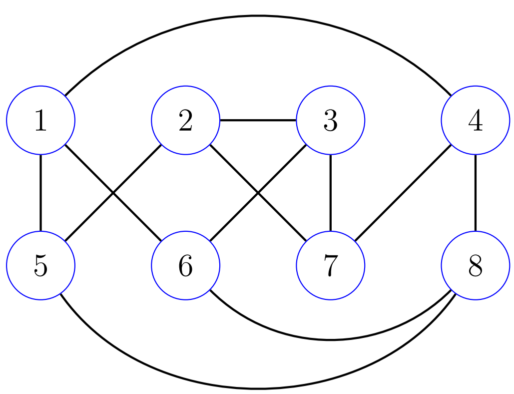
(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k) of `Dominating Set`, ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for (G, k).

(c) Provide size parameters that may be used to measure the size of an instance of (G, k) of `Dominating Set`.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

LO3. Recall the mapping reduction f : `HC` $\rightarrow$ `TSP` from `Hamilton Cycle` to `Traveling Salesperson` described in lecture.

(a) Given the `HC` instance G shown below, draw $f(G)$. Hint: $f(G)$ is comprised of two distinct entities.



(b) Verify that both G and $f(G)$ are both positive instances of their respective problems. Show work and explain.

LO4. Answer the following. Note: scoring 14 or more points counts for passing.

(a) Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

i. An instance of `Sum Avoidance` is a finite subset of integers S, and a target value t, and the problem is to decide if no subset of S has the property that its members sum to t.

ii. An instance $\mathcal{C}$ of `Balance SAT` is the same as an instance of `3SAT`, but now the question is whether or not there is an assignnment over the variables that i) assigns 1 to at least one literal from each clause, and ii) assigns 0 to at least one literal in each clause.

iii. An instance of `Composite` is an integer $n \geq 2$ and the problem is to decide if n is a composite number.

iv. An instance of `Bounded Half Clique` is a graph $G = (V, E)$ and the problem is to decide if the maximum clique in G is of a size that does not exceed $|V|/2$.

(b) Based on mapping reductions from the Mapping Reducibility and Computational Complexity lectures, the three polynomial-time mapping reductions $A \leq_m^p B$, $B \leq_m^p C$, and $C \leq_m^p$ `Set Partition` establish that `Set Partition` is an NP-complete problem. Provide the specific names of decision problems A, B, and C. (3 points each)

(c) Circle the correct answer. Which of the following is an appropriate certificate for verifying if an instance of `Traveling Salesperson` is positive? (9 points)

I. C is a subset of $|V|$ distinct edges.

II. C is a sequence of $|V|$ distinct edges.

III. C is a subset of $|V|$ distinct vertices.

IV. C is a sequence of $|E|$ distinct edges.

LO5. The **greatest common divisor** of two positive integers x and y, denoted $\gcd(x, y)$, is the greatest positive integer d for which d divides evenly into both x and y. Using any of the primitive recursive functions introduced in the Models of Computation lecture examples, show that $\gcd(x, y)$ is primitive recursive. Hint: iterate downward from a suitable upper bound rather than upward from 0 without misusing the iterator.

LO6. Solve the following problems.

(a) Compute the Gödel number for program $P = T(5, 2), Z(7), S(6), J(1, 2, 3)$. Write your answer as a sum of powers of two minus 1 (see part b for an example). Show all work.

(b) Provide the URM program P whose Gödel number equals

$$2^{21} + 2^{30} + 2^{98} + 2^{112} - 1.$$

Show all work.

Solutions

LO2. An instance of `Dominating Set` is a simple graph $G = (V, E)$ and a nonnegative integer $k \geq 0$, and the problem is to decide if there is a set $D \subseteq V$ of k vertices in G for which, for every vertex $u \in V - D$, there is vertex $v \in D$ for which $(u, v) \in E$. In other words, every vertex not in D is adjacent to some vertex in D.

(a) For a given instance (G, k) of `Dominating Set` describe a certificate in relation to (G, k).

Solution. D is a subset of V having size k.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k) of `Dominating Set`, ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for (G, k).

Solution.

Initialize function $\text{mark} : V \to \{0, 1\}$ as $\text{mark}(v) = 0$ for all $v \in V$.

For each $v \in D$,

  $\text{mark}(v) = 1$.

For each $e = (u, v) \in E$,

  If $u \in D$, then $\text{mark}(v) = 1$.

  If $v \in D$, then $\text{mark}(u) = 1$.

Return $\underset{v \in V}{\forall}(\text{mark}(v) = 1)$.

(c) Provide size parameters that may be used to measure the size of an instance of (G, k) of `Dominating Set`.

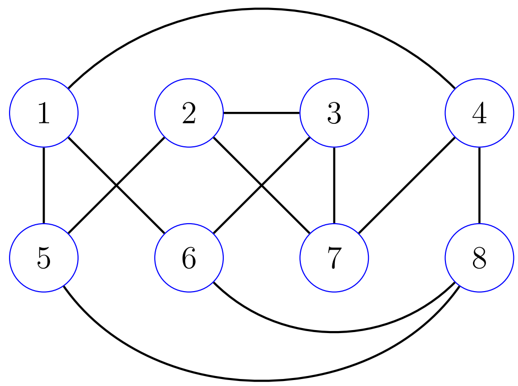
Solution. $m = |E|$, $n = |V|$.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

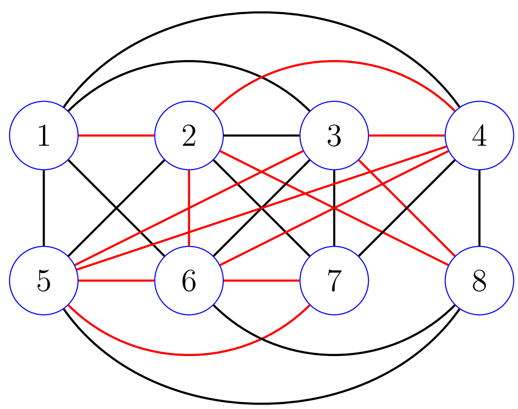
Solution. Marking all vertices in D can be done in $O(n)$ steps. When considering each $e = (u, v) \in E$, checking if either $u \in D$ or $v \in D$ can be done in $O(1)$ steps so long we create a hash table that holds the members of D. Creating such a table requires $O(n)$ steps and iterating over the edges thus takes $O(m)$ steps. Finally, checking that all vertices are marked can be done in $O(n)$ steps. Therefore, the verifier requires $O(m + n)$ steps.

LO3. Recall the mapping reduction $f : \texttt{HC} \to \texttt{TSP}$ from `Hamilton Cycle` to `Traveling Salesperson` described in lecture.

(a) Given the `HC` instance G shown below, draw $f(G)$. Hint: $f(G)$ is comprised of two distinct entities.



Solution. $f(G) = (G', k = 8)$ where the graph of G' is shown below. Note that G' is both complete and weighted, with black edges weighing 1 and red edges weighing 8.



(b) Verify that both G and $f(G)$ are both positive instances of their respective problems. Show work and explain.

Solution. G has the Hamilton Cycle $C = 1, 4, 8, 6, 3, 7, 2, 5, 1$ while the cost of this cycle in $f(G)$ is

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8 = k,$$

and so $f(G)$ is a positive instance of `TSP`.

LO4. Answer the following. Note: scoring 14 or more points counts for passing.

(a) Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

i. An instance of `Sum Avoidance` is a finite subset of integers S, and a target value t, and the problem is to decide if no subset of S has the property that its members sum to t.

ii. An instance $\mathcal{C}$ of `Balance SAT` is the same as an instance of `3SAT`, but now the question is whether or not there is an assignnment over the variables that i) assigns 1 to at least one literal from each clause, and ii) assigns 0 to at least one literal in each clause.

iii. An instance of `Composite` is an integer $n \geq 2$ and the problem is to decide if n is a composite number.

iv. An instance of `Bounded Half Clique` is a graph $G = (V, E)$ and the problem is to decide if the maximum clique in G is of a size that does not exceed $|V|/2$.

Solution. i) `Co-NP`, ii) `NP`, iii) `P`, iv) `Co-NP`

(b) Based on mapping reductions from the Mapping Reducibility and Computational Complexity lectures, the three polynomial-time mapping reductions $A \leq_m^p B$, $B \leq_m^p C$, and $C \leq_m^p$ `Set Partition` establish that `Set Partition` is an NP-complete problem. Provide the specific names of decision problems A, B, and C. (3 points each)

Solution. $A =$ `SAT`, $B =$ `3SAT`, $C =$ `Subset Sum`.

(c) Circle the correct answer. Which of the following is an appropriate certificate for verifying if an instance of `Traveling Salesperson` is positive? (9 points)

I. C is a subset of $|V|$ distinct edges.
II. C is a sequence of $|V|$ distinct edges.
III. C is a subset of $|V|$ distinct vertices.
IV. C is a sequence of $|E|$ distinct edges.

Solution. ii, since a Hamilton Cycle is a sequence of $|V|$ edges. The verifier can check if a) the edge sequence forms a valid cycle, and b) if the sum of the edge weights does not exceed the bound k on the tour cost.

LO5. The **greatest common divisor** of two positive integers x and y, denoted $\gcd(x, y)$, is the greatest positive integer d for which d divides evenly into both x and y. Using any of the primitive recursive functions introduced in the Models of Computation lecture examples, show that $\gcd(x, y)$ is primitive recursive. Hint: iterate downward from a suitable upper bound rather than upward from 0 without misusing the iterator.

Solution. We have

$$\text{Max}(x, y) - \underset{z \leq \text{Max}(x,y)}{\lambda} (\text{Div}(x, \text{Max}(x, y) - z) \wedge \text{Div}(y, \text{Max}(x, y) - z)).$$

LO6. Solve the following problems.

(a) Compute the Gödel number for program $P = T(5, 2), Z(7), S(6), J(1, 2, 3)$. Write your answer as a sum of powers of two minus 1 (see part b for an example). Show all work.

Solution.

$$\beta(T(5, 2)) = 4\pi(4, 1) + 2 = 4(47) + 2 = 190.$$

$$\beta(Z(7)) = 4(6) = 24.$$

$$\beta(S(6)) = 4(5) + 1 = 21.$$

$$\beta(J(1,2,3)) = 4\xi(0,1,2) + 3 = 4\pi(\pi(0,1),2) + 3 = 4\pi(2,2) + 3 = 4(19) + 3 = 79.$$

$$\gamma(P) = \tau(190, 24, 21, 79) = 2^{190} + 2^{215} + 2^{237} + 2^{317} - 1.$$

(b) Provide the URM program P whose Gödel number equals

$$2^{21} + 2^{30} + 2^{98} + 2^{112} - 1.$$

Show all work.

Solution. $P = S(6), Z(3), J(1,1,9), S(4)$.
For example, to get $J(1,1,9)$, we have $98 - 30 - 1 = 67$ and $67 \bmod 4 = 3$ which implies a Jump instruction. Also, $(67-3)/4 = 16$, and $\pi^{-1}(16) = (0,8)$ since $(16+1) = 2^0(2(8)+1)$.

Finally, $\pi^{-1}(0) = (0,0)$ since $(0+1) = 2^0(2(0)+1)$.

Therefore, we have $J(0+1, 0+1, 8+1) = J(1,1,9)$.