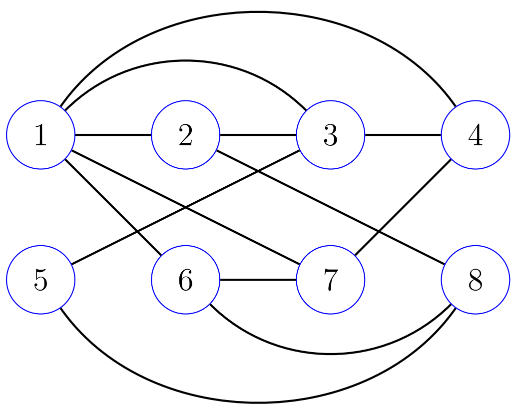
CECS 329, Learning Outcome Assessment 5, October 5th, Fall 2023, Dr. Ebert

Problems

LO1. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems.

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Maximum Independent Set (MIS)` optimization problem. Draw $f(G)$, where f is the mapping reduction from `MIS` to `Maximum Clique` provided in lecture.



(c) Verify that both G and $f(G)$ have the same solution. Hint: for both problems please assume that a "solution" is represented by a subset of vertices. Justify your answer.

LO2. An instance $(\mathcal{C}, m)$ of `Set Splitting` is a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_n\}$, where, for each $i = 1, \ldots, n$, $C_i \subseteq \{1, 2, \ldots, m\}$. The problem is to decide if $\{1, 2, \ldots, m\}$ can be partitioned into two sets A and B such that

(a) $A \cup B = \{1, 2, \ldots, m\}$

(b) $A \cap B = \emptyset$

(c) $C_i \cap A \neq \emptyset$, for all $i = 1, 2, \ldots, n$

(d) $C_i \cap B \neq \emptyset$, for all $i = 1, 2, \ldots, n$.

(a) For a given instance $(\mathcal{C}, m)$ of `Set Splitting` describe a certificate in relation to $(\mathcal{C}, m)$.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance $(\mathcal{C}, m)$ of `Set Splitting`, ii) a certificate for $(\mathcal{C}, m)$ as defined in part a, and decides if the certificate is valid for $(\mathcal{C}, m)$.

(c) Provide size parameters that may be used to measure the size of an instance of $(\mathcal{C}, m)$ of `Set Splitting`.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

LO3. Recall the mapping reduction f : `3SAT` $\rightarrow$ `Clique` from `3SAT` to `Clique` described in lecture. Given the `3SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_2, \overline{x}_3), (\overline{x}_1, x_2, \overline{x}_3), (x_1, x_2, x_3), (\overline{x}_1, \overline{x}_2, \overline{x}_3)\},$$

answer the following questions about $f(\mathcal{C})$.

(a) How many vertices and edges does the graph of $f(\mathcal{C})$ have? Show work and explain.

(b) What is the k value for $f(\mathcal{C})$? Explain.

(c) Is $f(\mathcal{C})$ a positive instance of `Clique`? Explain.

LO4. Answer the following. Note: scoring 14 or more points counts for passing.

(a) Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

i. An instance of `Reachability` is a simple graph $G = (V, E)$ and two vertices $u, v \in V$, and the problem is to decide if v is reachable from u, meaning there is a path from u to v.

ii. An instance of `Double-SAT` is a Boolean formula F and the problem is to decide if there are at least two different assignments that satisfy F. assignments.

iii. An instance of `Tautology` is a Boolean formula $F(x_1, \ldots, x_n)$, and the problem is to decide if F is a tautology, i.e. $F(\alpha) = 1$ for every possible assignment α over the variables $x_1, \ldots, x_n$.

iv. An instance of `2SAT` is a set $\mathcal{C}$ of binary clauses and the problem is to decide if there is an assignment α over the clause variables that forces each clause to evaluate to 1.

(b) What was the first problem in `NP` that was established as being `NP`-complete? (8 points)

   i. `Directed Hamilton Path`
   ii. `3SAT`
   iii. `SAT`
   iv. `Clique`

(c) Circle the correct answer. Which of the following is *not* a valid size parameter for the `Clique` decision problem. (9 points)

   i. m is the number of graph edges.
   ii. n is the number of graph vertices.
   iii. k is the desired clique size.
   iv. All of the above are valid size parameters.

LO5. Consider the function $\text{Digit}(x, y)$ which evaluates to the y th digit of x, where digit 0 denotes the ones place, digit 1 denotes the tens place, etc.. For example, $\text{Digit}(5862, 0) = 2$, $\text{Digit}(5832, 1) = 6$, and $\text{Digit}(5832, 4) = 0$. Note that $\text{Digit}(x, y)$ is primitive recursive.

Now consider the function $\text{Sevens}(x, y)$ which evaluates to x of but with each of digits 0 through y replaced by a 7. For example, $\text{Sevens}(5862, 0) = 5867$, $\text{Sevens}(5832, 1) = 5877$, and $\text{Sevens}(5832, 4) = 77,777$. Show that Sevens is primitive recursive. You may use the $\text{Digit}(x, y)$ function above as well as any of the primitive recursive functions from the lecture. Hint: using recursion is optional, but offers the most succinct solution.

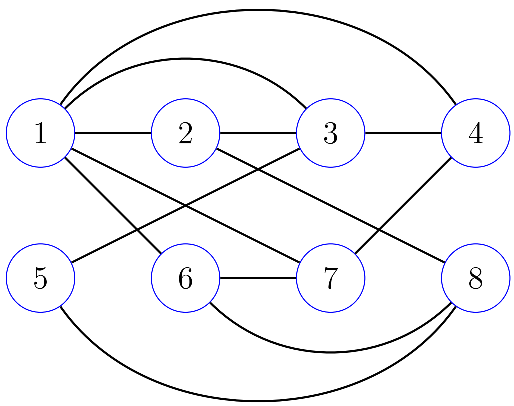
Solutions

LO1. Answer the following.

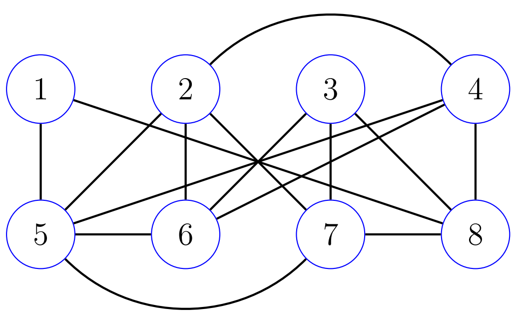
(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems.

Solution. See Definition 2.1 of Map Reducibility Lecture.

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Maximum Independent Set (MIS)` optimization problem. Draw $f(G)$, where f is the mapping reduction from `MIS` to `Maximum Clique` provided in lecture.



Solution. $f(G) = \overline{G}$ is shown below.



(c) Verify that both G and $f(G)$ have the same solution. Hint: for both problems please assume that a "solution" is represented by a subset of vertices. Justify your answer.

Solution. G's maximum independent set is $\{2, 4, 5, 6\}$ which is the maximum clique for $f(G) = \overline{G}$. Therefore, G and $\overline{G}$ have the same solution as is required for a mapping reduction.

LO2. An instance $(\mathcal{C}, m)$ of `Set Splitting` is a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_n\}$, where, for each $i = 1, \ldots, n$, $C_i \subseteq \{1, 2, \ldots, m\}$. The problem is to decide if $\{1, 2, \ldots, m\}$ can be partitioned into two sets A and B such that

(a) $A \cup B = \{1, 2, \ldots, m\}$

(b) $A \cap B = \emptyset$

(c) $C_i \cap A \neq \emptyset$, for all $i = 1, 2, \ldots, n$

(d) $C_i \cap B \neq \emptyset$, for all $i = 1, 2, \ldots, n$.

(a) For a given instance $(\mathcal{C}, m)$ of `Set Splitting` describe a certificate in relation to $(\mathcal{C}, m)$.

Solution. A certificate is a subset $A \subseteq \{1, 2, \ldots, m\}$.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance $(\mathcal{C}, m)$ of `Set Splitting`, ii) a certificate for $(\mathcal{C}, m)$ as defined in part a, and decides if the certificate is valid for $(\mathcal{C}, m)$.

Solution.

For each $C \in \mathcal{C}$,
  If $(A \cap C = \emptyset) \vee (A - C = \emptyset)$, then return 0.
Return 1.

(c) Provide size parameters that may be used to measure the size of an instance of $(\mathcal{C}, m)$ of `Set Splitting`.

Solution. $n = |\mathcal{C}|$ is the number of subsets, while m is a bound on the size of each subset.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

Solution. Using a hash table that stores the members of A, on can check the statement $A \cap C = \emptyset \vee A - C = \emptyset$ in $\mathrm{O}(|C|) = \mathrm{O}(m)$ steps. Thus, the loop (and hence verifier) requires $\mathrm{O}(mn)$ steps.

LO3. Recall the mapping reduction f : `3SAT` $\rightarrow$ `Clique` from `3SAT` to `Clique` described in lecture. Given the `3SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_2, \overline{x}_3), (\overline{x}_1, x_2, \overline{x}_3), (x_1, x_2, x_3), (\overline{x}_1, \overline{x}_2, \overline{x}_3)\},$$

answer the following questions about $f(\mathcal{C})$.

(a) How many vertices and edges does the graph of $f(\mathcal{C})$ have? Show work and explain.

Solution. The graph of $f(\mathcal{C})$ has $4 \times 3 = 12$ vertices and

$$7 + 7 + 8 + 7 + 8 + 6 = 43$$

edges. Every pair of distinct clause groups (c_i, c_j) produces up to 9 edges, but we must subtract one from this value whenver there is a literal $l \in c_i$ whose negation is in c_j. For example, for the (c_1, c_2) pair, the negations of x_1 and $\overline{x}_2$ appear in c_2, so we must subtract 2 from 9 to get 7 edges.

(b) What is the k value for $f(\mathcal{C})$? Explain.

Solution. We have $k = 4$ since $\mathcal{C}$ has four clauses and $\mathcal{C}$ is satisfiable iff $f(\mathcal{C})$ has a 4-clique, where the members of the 4-clique would be literals, one from each clause, each of which gets set to 1 by some assignment α that satisfies $\mathcal{C}$.

(c) Is $f(\mathcal{C})$ a positive instance of `Clique`? Explain.

Solution. Yes, since $\mathcal{C}$ is a positive instance of `3SAT` (it is satisfied by $\alpha = (1, 1, 0)$), $f(\mathcal{C})$ must also be a positive instance of `Clique`.

LO4. Answer the following. Note: scoring 14 or more points counts for passing.

(a) Write either P, NP, or co-NP next to each of the following decision problems in terms of which one best characterizes the complexity of the problem. Note: there is only one best answer for each. Two points each.

i. An instance of `Reachability` is a simple graph $G = (V, E)$ and two vertices $u, v \in V$, and the problem is to decide if v is reachable from u, meaning there is a path from u to v.

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iv. An instance of `2SAT` is a set $\mathcal{C}$ of binary clauses and the problem is to decide if there is an assignment α over the clause variables that forces each clause to evaluate to 1.

Solution. i: P, ii: NP, iii: co-NP, iv: P

(b) What was the first problem in NP that was established as being NP-complete? (8 points)

i. `Directed Hamilton Path`
ii. `3SAT`
iii. `SAT`
iv. `Clique`

Solution. iii (this is Cook's Theorem).

(c) Circle the correct answer. Which of the following is *not* a valid size parameter for the `Clique` decision problem. (9 points)

i. m is the number of graph edges.
ii. n is the number of graph vertices.
iii. k is the desired clique size.
iv. All of the above are valid size parameters.

Solution. iii since k does not always reflect the size of a problem instance. For example, graph G could have one million vertices and 1 billion edges, yet $k = 5$ (does G have a 5-clique?).

LO5. Consider the function $\text{Digit}(x, y)$ which evaluates to the y th digit of x, where digit 0 denotes the ones place, digit 1 denotes the tens place, etc.. For example, $\text{Digit}(5862, 0) = 2$, $\text{Digit}(5832, 1) = 6$, and $\text{Digit}(5832, 4) = 0$. Note that $\text{Digit}(x, y)$ is primitive recursive.

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Solution. Non-recursive solution.

$$\text{Sevens}(x, y) = \sum_{z \le y} 7 \cdot 10^z + \sum_{z=y+1}^{x} \text{Digit}(x, z) \cdot 10^z,$$

where $\sum_{z=y+1}^{x}$ is primitive recursive since

$$\sum_{z=y+1}^{x} = \sum_{z=0}^{x} - \sum_{z \le y}$$

is the difference between two summations that both begin with index 0.

Recursive Solution.

Base Case: $\text{Sevens}(x, 0) = x - \text{Digit}(x, 0) + 7$.

Recursive Case: $\text{Sevens}(x, y + 1) = \text{Sevens}(x, y) - \text{Digit}(x, y + 1) \cdot 10^{y+1} + 7 \cdot 10^{y+1}$.