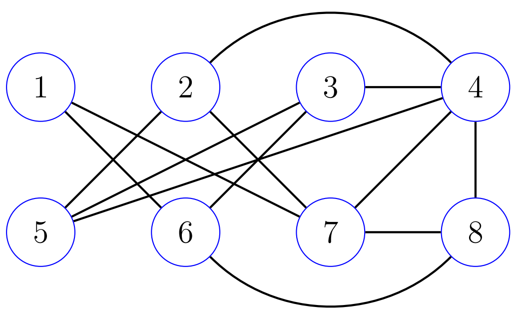
CECS 329, Learning Outcome Assessment 3, September 14th, Fall 2023, Dr. Ebert

Problems

LO1. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems.

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Maximum Independent Set (MIS)` optimization problem. Draw $f(G)$, where f is the mapping reduction from `MIS` to `Maximum Clique` provided in lecture.



(c) Verify that f is valid for input G in the sense that both G and $f(G)$ have the same solution. Hint: for both problems please assume that a "solution" is represented by a subset of vertices.

LO2. Recall that an instance of the `Vertex Cover` decision problem is a pair (G, k), where $G = (V, E)$ is a simple graph, $k \geq 0$ is an integer, and the problem is to decide if G has a **vertex cover** of size k, i.e. a set $C \subseteq V$ for which every edge $e \in E$ is incident with at least one vertex in C.

(a) For a given instance (G, k) of `Vertex Cover` describe a certificate in relation to (G, k).

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k), ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for G.

(c) Provide size parameters that may be used to measure the size of an instance of `Vertex Cover`.

(d) Use the size parameters from part c to describe the running time of your verifier from part b. Defend your answer in relation to the algorithm you provided for the verifier.

LO3. Recall the mapping reduction from `SAT` to `3SAT` described in lecture.

(a) Given the `SAT` instance $F(x_1, x_2, x_3) = \overline{x}_1 \vee (x_2 \wedge \overline{x}_3)$, draw its parse tree and provide the associated Boolean formula G that is satisfiability equivalent to F and serves as the beginning step of the reduction. Hint: formula G introduces y-variables.

(b) Rewrite formula G by making use of the logical identity

$$(P \leftrightarrow Q) \Leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$$

(c) Rewrite the formula from part b by making use of the logical identity

$$(P \rightarrow Q) \Leftrightarrow (\overline{P} \vee Q).$$

(d) Rewrite the formula from part c by performing one or more applications of De Morgan's rule.

(e) Rewrite the formula from part d by performing one or more applications of the distributive rule in order to obtain an AND of OR's. Then convert the AND of OR's to an AND of ternary (i.e. three) OR's and use `3SAT` notation to complete the reduction.

Solutions

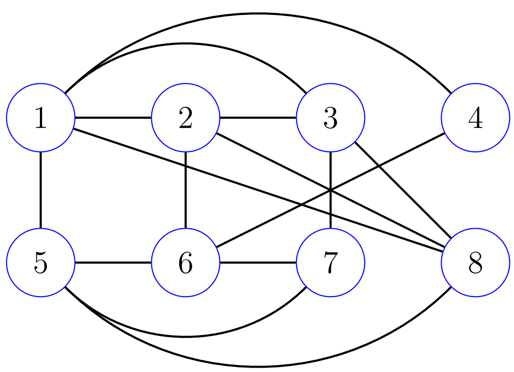
LO1. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems.

Solution. See Definition 2.1 of Map Reducibility Lecture.

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Maximum Independent Set (MIS)` optimization problem. Draw $f(G)$, where f is the mapping reduction from `MIS` to `Maximum Clique` provided in lecture.

Solution. $f(G) = \overline{G}$ is shown below.



Solution. G's maximum independent set is $\{1, 2, 3, 8\}$ which is the maximum clique for $f(G) = \overline{G}$. Therefore, G and $\overline{G}$ have the same solution as is required for a mapping reduction.

LO2. Recall that an instance of the `Vertex Cover` decision problem is a pair (G, k), where $G = (V, E)$ is a simple graph, $k \geq 0$ is an integer, and the problem is to decide if G has a **vertex cover** of size k, i.e. a set $C \subseteq V$ for which every edge $e \in E$ is incident with at least one vertex in C.

(a) For a given instance (G, k) of `Vertex Cover` describe a certificate in relation to G.

Solution. A certificate for (G, k) is a subset C of k vertices of G.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k), ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for G. **Solution.**

For each $e = (a, b) \in E$,

If $a \notin C \wedge b \notin C$, then return 0.

Return 1.

(c) Provide size parameters that may be used the measure the size of an instance of `Graph Matching`.

Solution. $m = |E|$, $n = |V|$.

(d) The verifier must consider each edge e of G and check that at least one vertex incident with e is in C. The check can be done in constant time with the help of a hash table that stores the members of C. Therefore, the verifier requires $O(m)$ steps.

LO3. Recall the mapping reduction from `SAT` to `3SAT` described in lecture.

(a) Given the `SAT` instance $F(x_1, x_2, x_3) = \overline{x}_1 \vee (x_2 \wedge \overline{x}_3)$, draw its parse tree and provide the associated Boolean formula G that is satisfiability equivalent to F and serves as the beginning step of the reduction. Hint: formula G introduces y-variables.

Solution.

$$y_1 \wedge (y_1 \leftrightarrow (\overline{x}_1 \vee y_2)) \wedge (y_2 \leftrightarrow (x_2 \wedge \overline{x}_3)).$$

(b) Rewrite formula G by making use of the logical identity

$$(P \leftrightarrow Q) \Leftrightarrow [(P \to Q) \wedge (Q \to P)]$$

Solution.

$$y_1 \wedge (y_1 \to (\overline{x}_1 \vee y_2)) \wedge ((\overline{x}_1 \vee y_2) \to y_1) \wedge (y_2 \to (x_2 \wedge \overline{x}_3)) \wedge ((x_2 \wedge \overline{x}_3) \to y_2).$$

(c) Rewrite the formula from part b by making use of the logical identity

$$(P \to Q) \Leftrightarrow (\overline{P} \vee Q).$$

Solution.

$$y_1 \wedge (\overline{y}_1 \vee (\overline{x}_1 \vee y_2)) \wedge (\overline{(\overline{x}_1 \vee y_2)} \vee y_1) \wedge (\overline{y}_2 \vee (x_2 \wedge \overline{x}_3)) \wedge (\overline{(x_2 \wedge \overline{x}_3)} \vee y_2).$$

(d) Rewrite the formula from part c by performing one or more applications of De Morgan's rule.

Solution.

$$y_1 \wedge (\overline{y}_1 \vee (\overline{x}_1 \vee y_2)) \wedge ((x_1 \wedge \overline{y}_2) \vee y_1) \wedge (\overline{y}_2 \vee (x_2 \wedge \overline{x}_3)) \wedge ((\overline{x}_2 \vee x_3) \vee y_2).$$

(e) Rewrite the formula from part d by performing one or more applications of the distributive rule in order to obtain an AND of OR's. Then convert the AND of OR's to an AND of ternary (i.e. three) OR's and use `3SAT` notation to complete the reduction.

Solution.

$$\{(y_1, y_1, y_1), (\overline{y}_1, \overline{x}_1, y_2), (x_1, y_1, y_1), (\overline{y}_2, y_1, y_1), (\overline{y}_2, x_2, x_2), (\overline{y}_2, \overline{x}_3, \overline{x}_3), (\overline{x}_2, x_3, y_2)\}.$$