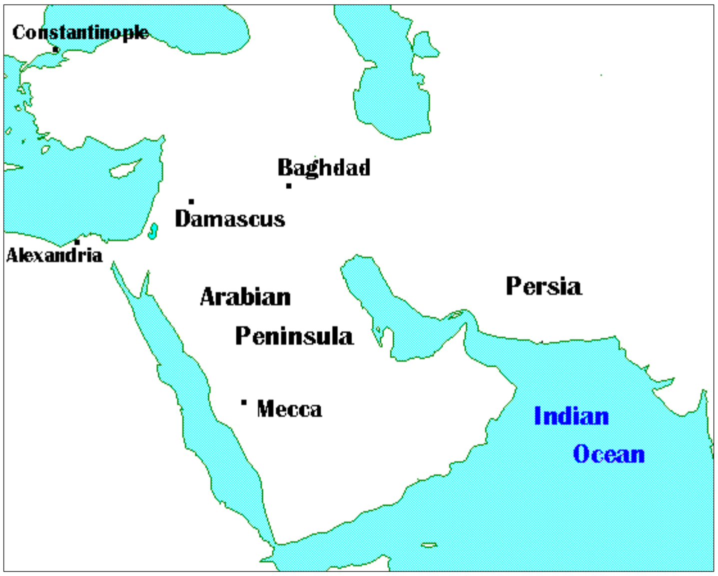
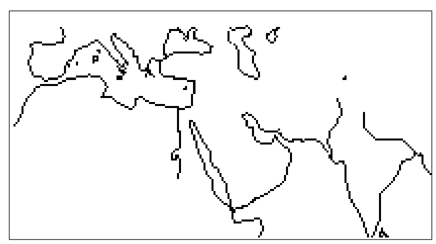
Chapter 9
Islam

Algebra is generous, she often gives more than is asked of her—*D'Alembert*



Mohammed, the prophet died in 632 AD. By the year 750, his influence, through both political and religious conversion, had spread from China to Spain. Although the movement was led by people from the Arabian Peninsula, it soon involved a multitude of nations, languages, races and ethnicities.

After the fall of Alexandria, and Northern India, the Muslims came in contact with the mathematical legacy of Greece, and India. For several crucial centuries, under the leadership of enlightened Caliphs, in their magnificent capitals, first **Damascus**, then **Baghdad**, and other cities such as **Cairo**, **Samarkand** and **Córdova**, spreading in three continents, the Arabs and other Muslims were the holders, disbursers and originators of most of the intellectual activity of the period. Baghdad became for a while the new Alexandria with a wonderful school: the **House of Wisdom** built in the early part of the ninth century by the Caliph Al-Mamum.



Vast energy was spent in the translation and additional commentary of most of the works that we have discussed during the Greek chapters—and lest we forget, Ptolemy's work is still known by the name given by the Arab conquerors of Alexandria. But the Arabs will not only translate and annotate, but also synthesize, clarify, and make crucial additions, as we will see below. They affected all branches of mathematics and science in their pursuits, from geometry to trigonometry to astronomy to geography to arithmetic to number theory to the creation of a more systematic and powerful algebra.

In the early part of the thirteenth century, the Mongols, led by Genghis Khan, ruthlessly invaded Persia and other parts of the Muslim world. But even before the Mongols then, extended contact between the Muslim world and the Chinese civilization had occurred. To cite just one example, the technology of paper came to Western Europe via this route, Chinese-Arab-Spanish (and Sicilian) soon after the Arabs reached China in 751.

Highlights of Arab and Islamic History

Date	Event
570	Prophet Mohammed is born in Mecca.
622	The Hegira: Mohammed flees Mecca for Medina—start of the Muslim calendar.
632	Death of Prophet Mohammed—Sunnis and Shiites argument develops.
642	Arabs conquer Persia and Syria led by Caliph Omar.
661-750	The Omayyad dynasty rules, Damascus is their capital.
711	Most of Spain is conquered by armies from North Africa. First Muslim state in India is established by Omayyads.
732	The franks stop Muslim invasion by winning the battle of Poitiers.
751	Islam reaches China.
762	Baghdad becomes capital of the Abbasid caliphate.
786-809	Harun Al-Rashid rules a united empire as the fifth Abbasid caliph.
813-833	Caliph Al-Mamum founds House of Wisdom.
992	The Ghaznavids form an independent state in Persia, and lead attacks against India.
1,071	Muslim Seljuk Turks conquer Persia and defeat Byzantine army.
1,085	Christians capture Toledo and its library.
1096	First Crusade starts.
1,167	Birth of Genghis Khan in Mongolia.
1,187	Saladin captures Jerusalem.
1,219	Mongols attack Persia and Turkey.
1,400	Damascus is devastated by the Mongols led by Tamurlane.

From the Western European point of view, other Arab intellectual centers also play a very meaningful role in the history of our subject. Some would consider the fall of **Toledo**, and its library, with its immense holding of texts, at the end of the eleventh century as pivotal in the resurgence of mathematical activity in Europe. Many translators came to Toledo and started the slow translation process of the Greek and Arab texts into vernacular European languages.

Islamic mathematics last roughly from 750 to 1,450. In that period they finished the development of the decimal system, together with the algorithms for most of the arithmetical operations: addition, subtraction, multiplication, division and extraction of roots. On the last topic, they not only improved the extraction of square and cube roots, they also developed an algorithm for the extraction of fifth roots!

Islamic mathematicians also contributed to the development of geometry and to the numerical solution of equations of higher degree than two. They began the consistent use of all six of our current trigonometric functions, going beyond the Hindu's sine and cosine. **Al-Tusi** from the 13th century would make trigonometry a mathematical science rather than a branch of astronomy, and he would clearly state laws such as the **Law of Sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In number theory, the Pythagorean and Diophantine tradition of perfect and amicable numbers was continued. A number is **perfect** if it equals the sum of its proper divisors, for example $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are both perfect. Islamic mathematicians of the 10th century (**al-Baghadadi**) would correctly claim that

> **He who affirms that there is only one perfect number in each power of 10 is wrong; there is no perfect number between ten thousand and one hundred thousand. He who affirms that all perfect numbers end with the figure 6 or 8 are right.**

This was in response to some false assertions made by Alexandrian mathematicians of the first century.

Two numbers are said to be **amicable** if the sum of the divisors of each equals the other number. From Pythagorean times, 220 and 284 had been known to be amicable since the divisors of 284 add up to 220, $1 + 2 + 4 + 71 + 142 = 220$, and, vice versa, the divisors of 220 add up to 284: $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$.

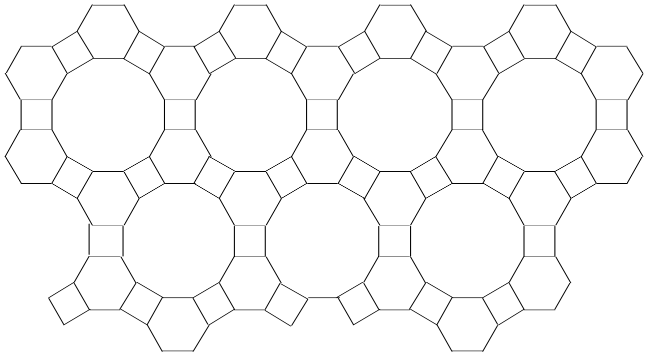
From the 9th century, **Thabit** would claim that

> For $n > 1$, let $p_n = 3 \times 2^n - 1$ and $q_n = 9 \times 2^{2n-1} - 1$. If p_{n-1}, p_n and q_n are prime numbers, then $a = 2^n p_{n-1} p_n$ and $b = 2^n q_n$ are amicable numbers.

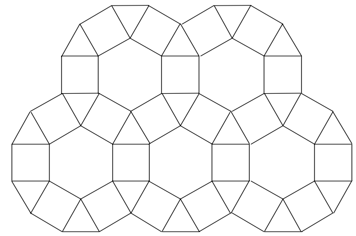
Thabit produced a new pair by using his claim when $n = 4$, so $p_3 = 23$, $p_4 = 47$ and $q_4 = 1151$, so $a = 17296$ and $b = 18416$.

They devoted great energy to both astronomy and geography. The **Almagest** would be studied and annotated by many authors during many centuries. Of particular interest was the calculation of the direction of Mecca as a function of terrestrial latitude and longitude since pilgrims attempted to visit the holy city from all over the world.

And in medicine, the physicians from Islam were superior to any other as exemplified by **Ibn Sina**, who is known in the West as **Avicenna**. He is considered to have written one of the most important books in the history of medicine, **The Canon of Medicine**. But he also wrote on mathematics, other sciences, astronomy and music.

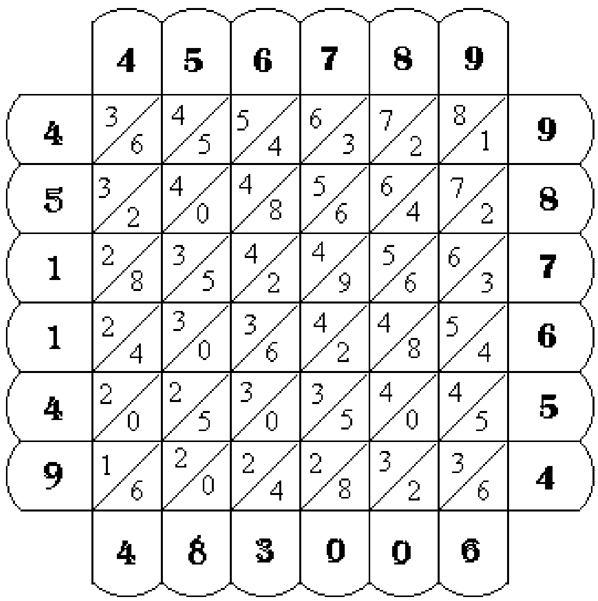


As a tenet of religious faith, humanistic representations of God are forbidden, hence, Muslims decorated their mosques with extraordinary abstract tilings, and rarely have such tessellations of the plane been more elaborate or beautiful. Naturally, such considerations lead to serious mathematical activity. For example, in one of their most famous buildings, the **Alhambra** in Granada (Spain), one can find all possible different symmetry tessellations of the plane.

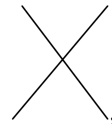
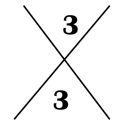


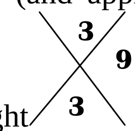
They had several multiplication and division algorithms, and notably the two of them we will discuss had geometric tilings in their background.

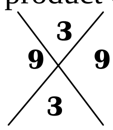
One of their most popular multiplication algorithms was called the **gelosia** or **lattice** method. The method came to Western Europe via **al-Khwarizmi**'s book (see below), and very soon in Italy, this method was extremely popular, and most early texts have examples of it. The example we give below is actually from a Florentine text from 1430. It involves the multiplication of $\mathbf{456{,}789} \times \mathbf{987{,}654}$.



and we readily read the answer
451,149,483,006.

We could use this example to illustrate another arithmetical contribution of the times, the method of **casting-out nines** which was taught in school as late as the 1950's. It was an effective way to test whether an error had occurred during a multiplication. The key idea was the Vedic transformation we mentioned in the previous chapter. Recall that this involved the consecutive addition of a number until a single digit remained. The rule was simple. If one is to multiply two numbers, then the Vedic transformation of their product is the same as the product of their Vedic transformation, itself transformed. One would do the following: take the numbers we are multiplying, $456{,}789 \times 987{,}654$, and write their Vedic transformations in the top and the bottom of , and we get 

since the Vedic transformation of 456,789 is $456789 \mapsto 39 \mapsto 12 \mapsto 3$ and similarly for the other factor. Multiply these two numbers (and apply the Vedic transformation if necessary) and write the result on the right 

, and finally, do the Vedic transformation to the product of the two numbers and write it on the left, the two numbers should be the same: 

which they are since $451{,}149{,}483{,}006 \mapsto 45 \mapsto 9$, which makes us happy! In modern times we would refer to this process as doing arithmetic modulo 9.

We now give an example of division. The method is called the galleon method, and it is a form of long division. Once again, the Arab influence on arithmetic is exemplified by the fact that the first printed occurrence of long division in the Americas occurred in a Spanish colony, New Spain (or Mexico) in the 16^{th} century. The reason for the name galleon is from the shape of the finished table that resembles a ship or galleon.

The table represents the division $65,284 \div 594$ with the result 109 with remainder 538.

	5						
~~1~~	~~6~~	8					
~~6~~	~~5~~	~~2~~	8	4	**1**		
5	9	4					

	5						
~~1~~	~~6~~	8					
~~6~~	~~5~~	~~2~~	8	4	**1**	**0**	
~~5~~	~~9~~	~~4~~	4				
	5	9					

	5						
~~1~~	~~6~~	8					
~~6~~	~~5~~	~~2~~	8	4	**1**	**0**	**9**
~~5~~	~~9~~	~~4~~	~~4~~	4			
	~~5~~	~~9~~	9				
		5					

	1						
	~~5~~	3					
~~1~~	~~6~~	~~8~~					
~~6~~	~~5~~	~~2~~	8	4	**1**	**0**	**9**
~~5~~	~~9~~	~~4~~	~~4~~	4			
	~~5~~	~~9~~	9				
		5					

	~~1~~	5					
	~~5~~	~~3~~					
~~1~~	~~6~~	~~8~~	7				
~~6~~	~~5~~	~~2~~	~~8~~	4	**1**	**0**	**9**
~~5~~	~~9~~	~~4~~	~~4~~	4			
	~~5~~	~~9~~	~~9~~				
		~~5~~					

	~~1~~	5					
	~~5~~	~~3~~	3				
~~1~~	~~6~~	~~8~~	~~7~~	8			
~~6~~	~~5~~	~~2~~	~~8~~	~~4~~	**1**	**0**	**9**
~~5~~	~~9~~	~~4~~	~~4~~	~~4~~			
	~~5~~	~~9~~	~~9~~				
		~~5~~					

Again, the importance of the ability to perform both multiplication and division quickly and efficiently cannot be over stressed. There is the tale of the fifteenth century German merchant who asked a German scholar where he should send his child for higher learning, and the scholar replied that if all he needed was for the child to perform addition and subtraction, then German universities would suffice. But that if the merchant wanted the child to do multiplication and division, he best send his child to Italy.

We will discuss three Islamic mathematicians, one from the early period, one from the middle and one from the late: **Abu Ja'far Muhammad ibn Musa al-Khwarizmi**, **Omar Khayyam** and **Jamshid al-Kashi.**

Al-Khwarizmi

Details of **al-Khwarizmi's** life are scarce. We know he lived from about 780 until 850. Some authors are confident he was born in Baghdad, but some believe he was originally from Central Asia, near the Aral Sea, in the outreaches of Persia. His mathematical contributions were made while at the **House of Wisdom** in Baghdad, and his works were dedicated to the Caliph al-Mamum.

He was momentous in establishing the decimal tradition in the Islamic world, and thus also in Europe. His lasting influence is still felt in two very common words in the mathematical lexicon.

First, he wrote a book on arithmetic called **The Book of Addition and Subtraction According to the Hindu Calculation.** When this book came to Western Europe, his name is going to be associated with the idea of knowing how to do arithmetic. With an Italian pronunciation, his name became **alcorismi**, from which our word **algorithm** comes from.

Second, another of his books is called in the original Arab: **Hisab al-jabr wa'l-muqabala, which** means **The Book of Restoring and Balancing**. It is this title that gave us the word **algebra**, and, naturally, this book is pivotal in the history of the subject.

Not as symbolic as present-day algebra, this early algebra uses mostly words. What we usually denote by x, it was called **root** or **thing**. A constant in an equation was **number**, what we call x^2 was referred to as **mal.**

Al-Khwarizmi considers mainly linear and quadratic equations and he identifies six types (three more than Euclid as he allows 0 as a coefficient):

squares equal to roots $x^2 = 3x$	squares equal to numbers $x^2 = 9$	roots equal to numbers $3x = 15$
squares and roots equal to numbers $x^2 + 10x = 39$	squares and numbers equal roots $x^2 + 21 = 10x$	roots and numbers equal to squares $3x + 4 = x^2$

Restoring and balancing refer to transforming an equation into one of these 6 forms. For example, $x^2 = 40x - 4x^2$ is **al-jabr** into $5x^2 = 40x$ since we removed a subtraction; while the equation $50 + 3x + x^2 = 29 + 10x$ is **al-muqabala** first into $21 + 3x + x^2 = 10x$ and then **al-muqabala** again into $21 + x^2 = 7x$.

One of the problems from al-Khwarizmi's book is:

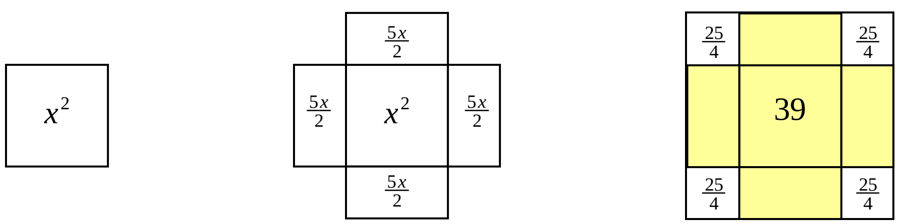
Solve mal and 10 root equals 39.

His solution is given in

> … a mal and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the mal which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square.

This paragraph represents syncopated algebra where common words, as opposed to symbols, are used to express the algebraic manipulations. The symbolic algebra will not be born until the Renaissance in Western Europe. The main contributions will be Italian and German as we will see in that chapter.

Al-Khwarizmi then gives a geometric justification for the argument by completing the square:



Naturally, nowadays, we would simply proceed by

$$x^2 + 10x = 39 \qquad (x+5)^2 = 39 + 25 = 64 \qquad x + 5 = 8 \qquad x = 3.$$

The negative root $x = -13$ is ignored.

Al-Khwarizmi further discusses (all in words, naturally) the multiplication of such expression such as $(2x+3)(4x+5)$.

One of the roles mathematics played in Islam was in the computation of inheritances. There were specific laws that dealt with inheritance. Some of the guidelines were as follows:

- If a wife died, the husband inherited one-fourth of the state, and the rest is divided among the children.
- A son receives twice what a daughter receives.
- A stranger cannot receive more than one-third of the state without approval of the other heirs.
- If a share is left to a stranger, this is taken off the top—before any distribution is made.

So another problem we encounter in al-Khwarizmi's Algebra is:

> A woman dies leaving a husband, a son and three daughters. She also leaves a bequest consisting of $\frac{1}{8}+\frac{1}{7}$ of her estate to a stranger. Calculate the shares of her estate that go to each of her beneficiaries.

The stranger then receives, $\frac{1}{8}+\frac{1}{7}=\frac{15}{56}$. Leaving $\frac{41}{56}$ of the estate to be shared by husband and children. The husband receives then one-fourth of $\frac{41}{56}$, which equals $\frac{41}{224}$. This leaves $\frac{123}{224}$ of the estate. We divide this into 5 equal parts—giving the denominator 1120, since it has to be shared in the ratio 2:1:1:1. Final allocations are then:

Stranger	$\frac{300}{1120}$
Husband	$\frac{205}{1120}$
Son	$\frac{246}{1120}$
Each Daughter	$\frac{123}{1120}$

Omar Khayyam

Omar Khayyam (c.1048-1131) is widely known in world literature for his poetic masterpiece, the **Rubaiyat**. Not as well known is the fact that he was an excellent mathematician. He lived his life in what was then Persia during very turbulent times. He served the Seljuk Turk ruler of the region at the observatory in Esfahan for more than 18 years, which was one of his most productive periods. It was at the observatory that Khayyam participated in a reform of the calendar. At one time he suggested that instead of every fourth year being a leap year, it should be eight years out of every 33 that ought to be leap years, and thus the length of the years would be 365.2424 which is closer to the astronomical year (365.2422) than the Julian year of 365.25, but also than the presently used Gregorian, 365.2425. He accurately measured the length of the year to be 365.24219858156 days. Naturally, there is no necessarily straightforward way to select 8 years out of 33 of them.

Khayyam had a truly wide spread scope of interests and talents as evidenced by both poetry and mathematics. But even within mathematics, he studied the foundations of Euclidean geometry, in particular the fifth postulate of Euclid, as well as the nature of the ratio of two numbers, including Eudoxus' definition given in Euclid.

Khayyam's name is closely associated with what was perhaps the first general study of

cubic equations—a very popular topic in the early fifteen hundreds in Italy. Partly in the desire for the duplication of the cube, and, partly out of general intellectual curiosity, Omar Khayyam set out to classify the types of cubic equations that there were. Many centuries later, Newton would be interested in cubic equations in two variables, and in the late 20th century, cubic equations would be used in the solution of **Fermat's last theorem**.

We have seen that the quadratic equation: $x^2 + ax + b = 0$ appeared very early in history, and was completely solved possibly as far back as 4,000 years ago. We looked at how al-Khwarizmi had given complete solutions for all 6 types of quadratic equations. The cubic emerges early in history too; for example the duplication of the cube is nothing else but a solution to the equation $x^3 = 2$.

As with the quadratic, we, in modern times, think of only one type of cubic equation:

$$x^3 + ax^2 + bx + c = 0.$$

But that is because we freely consider negative and positive numbers and zero as equivalent. Moreover, we completely ignore the nature of the roots, whether they are real or not. Khayyam naturally concentrated on the ones that had real solutions. Thus, $x^3 + 2x - 4 = 0$ and $x^3 - 2x + 4 = 0$ seem of similar type to our eyes, but to Omar Khayyam, they seem as of two different types:

$x^3 - 2x + 4 = 0$	$x^3 + 4 = 2x$	**Cube and numbers equal sides.**
$x^3 + 2x - 4 = 0$	$x^3 + 2x = 4$	**Cube and sides equal numbers.**

The last column indicates how Omar Khayyam would refer to the type of equation. In all, he classified 25 different types of cubic equations, and he gave methods for solving all of the ones that had had real (mainly positive) solutions.

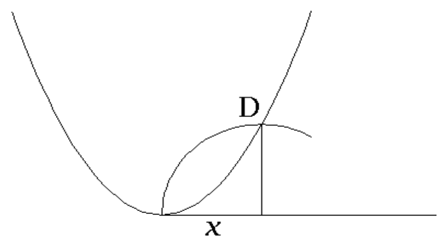
For eleven different types, he gave methods that involved only the Euclidean tools, straightedge and compass, but for others, he used conics to get the solution. He would be proven correct more than seven centuries later when he claimed that **straightedge and compass alone were not sufficient to solve all cubics**.

We end our discussion of Omar Khayyam with an example of one of his solutions:

Cube and sides equal numbers.	$x^3 + px = q$ **where p and q are positive numbers**
Take a so that it is the side whose square equals the number of roots.	$a = \sqrt{p}$
Let h be the side of a rectangular parallelepiped whose base is a^2 and whose volume is q.	$h = \dfrac{q}{p}$

Take a parabola whose vertex is **B**, axis **BZ** and parameter a, and place h perpendicular to **BZ** at **B**.	Consider the parabola $ay = x^2$, and find h in the $x-$axis.
On h as diameter describe a semicircle and let it cut the parabola at **D**.	Find the intersection **D** of the parabola with the circle $\left(x-\frac{h}{2}\right)^2 + y^2 = \left(\frac{h}{2}\right)^2$.
From **D**, drop **DE** perpendicular to h and the ordinate **DZ** perpendicular to **BZ**.	Find the coordinates of **D**
Then $\mathbf{DZ} = \mathbf{BE}$ and **BE** solves our equation	The $x-$coordinate of **D** is a solution.

The modern proof that the construction works is immediate: the equation of the semicircle is $x^2 - hx + y^2 = 0$, or equivalently, $y^2 = x(h-x)$, or also $a^2y^2 = a^2x(h-x)$. From the parabola, we know that $x^2 = ay$ so $x^4 = a^2y^2$. Equating we get $x^4 = a^2x(h-x)$, and canceling x, $x^3 = a^2(h-x) = p(h-x)$, thus $x^3 + px = ph = q$ and we have solved the equation.



Omar Khayyam in his construction respects at all times the homogeneous nature of the equation $x^3 + px = q$ in that p is an area and q is a volume, so the geometric spirit of the Greeks lives on. By the way, in order to build h, he needed a segment of length 1, and then he gave a relatively straightforward Euclidean construction of h.

Al-Kashi

As one may say that the art of Omar Khayyam was geometric algebra, one should state that the art of Jamshid al-Kashi was numerical algebra—he was a number-cruncher. Appropriately, he wrote a book called Calculator's Key. We do not know when al-Kashi was born, but we know that he was active from 1406-1429, when he passed away. He lived most of his productive years in Samarkand, in Central Asia, in what was then Persia. There he was the leading astronomer and mathematician of a large group of scientists assembled there by the ruler-scientist Ulugh Beg, a grandson of the Mongol conqueror, Tamurlane.

Al-Kashi improved on the algorithm for taking square roots that the Indus had developed, but more remarkably he gave a precise algorithm to compute the fifth root of a number. However, in our modern days of calculators, the fifth root algorithm is beyond our manual dexterity, and instead we will delve into an algorithm for taking square roots, which as mentioned above also was of al-Kashi's concern.

We view the algorithm in the concrete by looking at an example. As all similar root algorithms, **it computes the square root one digit at a time**. Clearly, the algorithm can be extended to pursue decimal digits (as many as we desire) as we will exemplify below.

1	2
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81

Before we proceed it would be convenient to have a table of the digits and their squares. We also need to observe what the powers of 10 have for their squares

Power	Square
10	100
100	10,000
1,000	1,000,000

Suppose now we want to compute $\sqrt{605,061,699}$. The first step is to realize is that we are taking the square root of 605 million plus. This will give us the square root of 605 thousands plus, so we need to start by taking the square root of 605. Clearly, the square root of 605 is a two-digit number: $10a+b$ where a and b are digits. Thus when we square we have $(10a+b)^2 = 100a^2 + 20ab + b^2$, and so $a^2 \leq 6$, and so $a = 2$.

That is exactly the starting point of the algorithm. Actually, more accurately, the algorithm starts by separating the number whose square root we are taking into blocks of two digits starting on the right (since the square of 10 is 100).

6,05,06,16,99	2
4	
2,05	

Thus we have found a, the first digit for the answer for the square root of 605. Subtract its square and the remainder together with next block, namely, 2,05 will be use to estimate b, the second digit. To do this, we double what we have obtained so far for our square root (only a 2 so far), and separate the last digit of our remainder (since remember the term is $20ab$), and

6,05,06,16,99	2
4	4
20,5	

Ask how many times does our doubled result goes into our remainder: $20 \div 4 = 5$, and so we attach the 5 behind the 4, and multiply it by 5, obtaining 225

6,05,06,16,99	2
4	45
205	5
	225

which is too big since our remainder is only 205, so our estimate $b = 5$ was too high, and here is one of the crucial ingredients to our algorithm, the **need to erase at times**. Thus, before this algorithm could be immensely popular, technological improvements such as paper and pencil, had to allow erasure. So we change our estimate to 4

6,05,06,16,99	2
4	44
205	4
176	176
29	

so our next digit is 4, we enter it, and double our result. We also bring the next block next to our remainder, and again separate the last digit

6,05,06,16,99	24	
4	24	48
205	4	
176	176	
290,6		

Again we ask $290 \div 48 = 6$, and we can estimate the next digit to be 6,

6,05,06,16,99	24	
4	24	486
205	4	6
176	176	2916
290,6		

but, as before, it is too high, so we have to change down again

6,05,06,16,99	24	
4	24	485
205	4	5
176	176	2425
2906		
2425		
481		

Thus we enter our new digit 5, double our result, and bring the next group of two digits, and separate the last digit:

6,05,06,16,99	245		
4	24	485	490
205	4	5	
176	176	2425	
2906			
2425			
4811,6			

and estimating $4811 \div 490 = 9$, and testing

6,05,06,16,99	245		
4	24	485	4909
205	4	5	9
176	176	2425	44181
2906			
2425			
48116			
44181			
3935			

so no erasure is necessary this time. Continuing,

6,05,06,16,99	2459			
4	24	485	4909	4918
205	4	5	9	
176	176	2425	44181	
2906				
2425				
48116				
44181				
39359,9				

and estimating $39357 \div 4918 = 8$, and testing

6,05,06,16,99	24598			
4	24	485	4909	49188
205	4	5	9	8
176	176	2425	44181	393504
2906				
2425				
48116				
44181				
393599				
393504				
95				

and we are done; our answer is 24,598 (we have a remainder).

Now we show how to compute decimal digits by adding blocks of 2 zeroes for each decimal digit we wanted.

We will proceed to do the first three decimals, without any dialog

6,05,06,16,99.00,00,00	24598.001						
4	24	485	4909	49188	491960	4919600	49196001
205	4	5	9	8	0	0	1
176	176	2425	44181	393504	0	0	49196001
2906							
2425							
48116							
44181							
393599							
393504							
950,0							
0							
95000,0							
0							
95000000							
49196001							
45803999							

So the answer to three decimals is 24,598.001.

Al-Kashi may have been one of the first ones to have used decimal notation for digits to the right of the decimal point. This notation will not be used in the West for another century. As a matter of fact, Fibonacci, who introduced Hindu-Arabic numerals to Italy in the thirteenth century, always used hexagesimal notation to express numbers.

Al-Kashi used both decimal and hexagesimal notation to give an approximation of 2π which is correct to 16 decimal places. Namely,

$$2\pi = 6;\ 16,\ 59,\ 28,\ 1,\ 34,\ 51,\ 46,\ 14,\ 50,$$

or, decimally,

$$2\pi = 6.2831853071795865.$$

He accomplished his approximation by using the half-angle formula 28 times, and being very careful with the error estimates. This is tantamount to inscribing and circumscribing polygons with 805,306,368 sides in a circle! But, as we mentioned above, he used numbers to do the estimation.

This represented a considerable improvement over the two digits of Archimedes. Also, it was considerably better than the approximation given in the fifth century AD by the Chinese mathematician **Zu Chongzhi** who gave the fraction $\frac{355}{113}$ as an approximation.

This is accurate to 6 digits. However, when Calculus was developed, much more powerful methods of approximation will be developed, methods that will supersede al-Kashi's approximation by tens of digits. Many distinguished mathematicians participate in this hunt—including Newton. In recent times, the first two billion digits of π have been computed using both mathematics and computers.