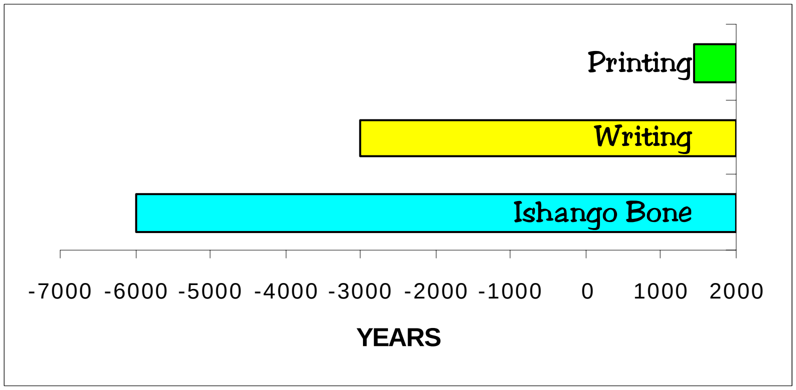
Chapter 1
Roots

I wish I knew as much as I thought I
knew 10 years ago—*Neugebauer*

As one begins to contemplate the history of mathematics, one begins to appreciate the vastness of its panorama. Mathematics, together with music and art, must be among the first intellectual pursuits of mankind. Mathematics certainly antecedes writing and the dawn of history—but even further, there is evidence that it antecedes language. In this chapter, ideas and notions that occurred early, in prehistory, before any written records that have survived, are discussed. Mathematics has been referred to as **the study of number, shape, arrangement, change and chance**. It is, certainly, the first two of these, **number** and **shape**, that occur earliest in the human experience. It is known, for example, that evidence exists of counting and tallying before writing.

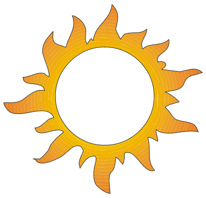


In Nigeria, Africa, a bone (with 29 scratches) was found, which indicates tallying activity as far back as the year 6,000 BC. It has been speculated that the **Ishango Bone**, as it is known by, represents the tallying of a lunar month. The picture on the right should help visualize how old this evidence is as we compare it with two other monumental events in the history of mankind, the advent of writing in the third millennium BC, and the invention of printing, approximately 500 years ago. Recently, even older evidence of counting has been found in Europe.

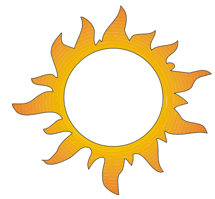
A little reflection will in fact make one realize that **measurement of both space and time must have provided early and powerful incentives for mathematical activity**. In fact, hunters if interested in hunting at night must have been soon interested in the motion of the moon, as darker nights made the hunting easier. Thus, one can speculate that the measurement of time led possibly to some of the first concepts of numbers. Or maybe, it was space, and the counting of children, or sheep, or soldiers that led to the idea of number. In any case, evidence exists of very early arithmetical activity, and it may even be true that symbols for numbers appeared before any other written language.

Since the measurement of time, in particular, the calendar, was one of the most universal stimulators of arithmetical activity, we will take a small detour and discuss some of the basic facts about it.

The Calendar

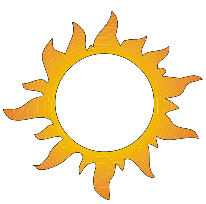



The **Sun** and the **Moon** have played a pivotal role in nearly all cultures. Most of the early units for the measurement of time are involved with at least one of these two very visible astral bodies. We will review these words one at a time in the order of ease of perception.



The **Day** is the most easily perceived as the motion of the Sun around the Earth. Thus it is Solar in origin, and there is a universal understanding of day. We have speculated before that the counting of days is one of the first origins of counting.

Next in ease of perception is the **Month**, which, as its name alone indicates, is Lunar in origin. The Month is the cycle of the Moon as it travels from Full to New and Full again. It is roughly $29\frac{1}{2}$ days. Many cultures have based their calendar on the month, and then, these are called **Lunar Calendars**. Most importantly, the **Mesopotamians** used such a calendar, and their influence persists today in both the **Jewish** and the **Arabic** calendars, but independently, the **Chinese** also use a lunar calendar. More will be said about Lunar Calendars once the most important of the words has been discussed.



The **Year** is more easily perceived in higher latitudes where the length of days has the higher variation. Nevertheless, the ancient **Egyptians**, as well as many other ancient cultures, developed a firm understanding of its duration. The importance of the year for them was the yearly, regular flooding of the Nile. With some thought, and considerable patience, one can estimate the length of the year by using a stick firmly planted, and each day when the Sun is at its highest, marking the shadow of the stick on the floor. If one does this regularly, one will see a pattern of shadows on the floor that which will eventually cycle—return to its starting point, and thus one would have calculated the length of the year, namely the duration of the long cycle of the sun around the earth from the ancient point of view.

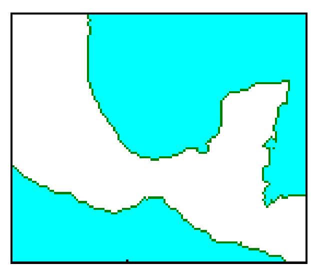
This would give the well-known 365 days without too much difficulty, but to approximate further takes significantly more energy and patience. We know now that a year is approximately $365\frac{1}{4}$ days. The **Egyptians** indeed were aware that the year was 365 days long—but for a long time did not have the quarter of a day fraction. A vast amount of computation and mathematics has been developed to deal with the approximations caused by the lengths of both the Month and the Year. For many centuries, the leading mathematicians in many cultures were the leading astronomers. This was the case in **India**, for example, as we will see below.

Of course, we have 12 months in a year because 12 is the closest one can come to fitting in $29\frac{1}{2}$ days into 365 days. The importance of the seasonal return of the weather and how to adjust it to the Lunar Calendar has been dealt with in a variety of ways. In some cultures an extra month is added to the year to balance the year, and hence some years will have 13 months in the **Jewish** and **Arabic** calendar.

The leap year adjustment goes back to **Julius Caesar**. By his time, approximately 50 BC, the seasons were so out of tempo that it was snowing in July (which was called Quintillius before him). Thus, he sought the counsel of scientists in Alexandria (which is in Egypt), and of **Sosigenes** in particular. Julius, then, in typical imperial fashion, introduced the addition, every four years, of an extra day in February (which was the last month of the year in the Roman calendar). It was then that the year changed from 365 days to $365\frac{1}{4}$ days. He also declared the correct day to be March 1 (the first day of the Roman calendar), so the months would be in harmony with the seasons. However, the decree caused that specific year to last more than 440 days! But then the calendar was correct for quite an extended period of time—for more than 1500 years!

However, by 1582, the calendar was again running out of tune, and it was adjusted again, this time by the Pope, **Gregory XIII**, and thus our calendar is the **Gregorian Calendar**. The Gregorian modification is simple. A year is actually slightly less than $365\frac{1}{4}$ days. And so not every fourth year should be a leap year, and thus by the Gregorian command, century years will not be leap years unless the first two digits are a multiple of four. Thus, the year 1700 was not a leap year, but the year 2000 was. This makes the duration of the year to be 365.2425 days, which is closer to the astronomically more accurate 365.2422 days, and it is accurate up to more than 3,000 years. But it will have to be adjusted in the future!

By the time of the Gregorian adjustment, 9 days had to be deleted from the calendar to correct it. The Gregorian calendar was not adopted in the United States until 1752 since it was a Catholic idea, and by then 11 days had to be deleted. **Perhaps it is not well known that George Washington was actually born on February 11, 1732?**



We can see that all this needed calculation leads to dexterity with numbers. Calendric considerations have played a role with developing notation to improve the computations among other things. We discuss briefly the **Mayans** of Southern Mexico and Guatemala.

The **Mayans** did not use the motions of the Moon as a foundation for their months. Their months had 20 days—and 20 was also their base. Base 20 is a rare occurrence since, as expected, the most common base among cultures is 10. Yet, one can conjecture that the hot climate led to such a basis, and, not coincidentally, cultures in West Africa, the **Yoruba**, for example, also used base 20.

The Mayans had an outstanding calendar and excellent number notation. They inherited ingredients for both from the Olmecs (the Aztecs inherited their calendar from the Olmecs also), but by the year 250 BC, the Mayans had already made significant improvements on both of them. In fact, some modern day authors give the Mayan calendar a duration of 365.2420 days, which is even closer to the current measurement that the Gregorian 365.2425.

The **Mayan** calendar consisted of two parts, the **Tzolkin** and the **Haab**. Their word for day was **kin**. The **Tzolkin** was a cycle of 260 days-which is close to the length of human gestation. It was divided into 13 groups of 20. The **Haab** consisted of 365 days divided into 18 months of 20 days each with an additional 5 days of omen and religious significance. They adjusted the calendar by using their very accurate placement of Venus in the sky. They were aware that Venus is both the morning star and the evening star—a fact that many cultures had no knowledge of.

Hence to the **Mayans**, who believed in the cyclical essence of nature, one of their cycles consisted of 18,980 days. This is the length of time it will take for both calendars to restart—in modern days one would notice that 18,980 is the least common multiple of 260 and 365. But they had other and longer cycles, one of over 1 million days. Eventually, they believed the world would end on **December 24, 2011**.

Since the Mayans could manipulate large numbers, one would expect them to have had a good number notation. Indeed they did. Their number system used two symbols: $\circ$ stood for 1 (**think of a finger perpendicular to one's eye**) and the symbol: | or — which stood for 5, (**think of a hand perpendicular to one's eye**). Thus to write 7, they would simply write $\underline{\circ\circ}$. They also used a positional notation, thus $\underline{\underline{\circ\circ\circ}}\ \ \underline{\circ\circ}$ stood for thirteen of the base, 20, plus seven units, in other words, it represented 267. However, they did not use the positional notation as consistently as we do. The third digit, instead of $20^2 = 20 \times 20 = 400$, represented the more calendrically correct $18 \times 20 = 360$. Thus, $\underline{\underline{\circ\circ\circ\circ}}\ \ \underline{\underline{\circ\circ\circ}}\ \ \underline{\circ\circ}$, stood for $14 \times 360 + 13 \times 20 + 7 = 5{,}307$. But, after that digit they would go back to the traditional times 20, so the fourth digit stood for $20 \times 18 \times 20$.

More surprisingly, the **Mayans** had a placeholder—they had a **zero**. Although we obtained ours from **India** (see the table below), we have to pay compliment to a rare achievement indeed, one that many cultures failed to reach. They used a shell-like symbol for 0. We will use ❄, although their glyphs were much more elegant. For the Mayans, then, $\underline{\underline{\circ\circ\circ}}$ ❄ $\underline{\circ\circ}$ stood for

$$13 \times 360 + 7 = 4{,}687 .$$

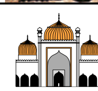
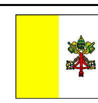
We have just begun to understand the subtleties of Mayan mathematics since their hieroglyphics were not decoded until the 1980's. One of the reasons for the delay in deciphering was the lack of Mayan documents. Most of them were destroyed by Bishop Landa shortly after the conquest of Yucatan.

The table on the right illustrates the ubiquitous search for the length of the year by every culture, and at all times, from ancient to modern.

The Roman influence on the calendar is much more pervasive than just the addition of leap years.

In fact, the names of the months were originally given to us by the Romans. Of course, the month of **March** was named after their most powerful god, Mars, and it indicated the beginning of their year. Other months named after some gods and deities in the Roman Pantheon are **April**, **May**, **June** and **January**. The last month of their year, **February**, was considered a somber period of self-examination and reflection.

The Length of the Year

	# of days	The number of days in the year in		Where	When
❶	365	most ancient solar calendars		Egypt	Ancient
❷	365.25 or $365\frac{1}{4}$	the Julian Calendar		Rome-Alexandria	First Century BC
❸	365.244407 or $365\frac{743}{3040}$	the Chinese Calendar		China	Eighth Century
❹	365.2420	the Mayan Calendar		Mexico-Guatemala	Ninth Century
❺	365.2424… or $365\frac{8}{33}$	Omar Khayyam's Calendar		Persia	Eleventh Century
❻	365.2425 or $365\frac{97}{400}$	the Gregorian Calendar		Rome	Sixteenth Century
❼	365.242193	today's measurements		The World	Twentieth Century

July and **August** are named after Julius and **Augustus** (his avenger and successor), but the last four carry number names **September**, 7; **October**, 8; **November**, 9; and **December**, 10. One may notice that the numbers are not correct for us, but they are indeed correct in the Roman count—in fact, if the year were to start in March, December would be the tenth month.

Of the remaining units of measurement of time, **week**, **hour**, **minute** and **second**, most were created for arithmetical convenience. The **hour** was a twelfth of the night and a twelfth of the day. The reasons for 12 are not known. Perhaps the Egyptians chose 12 because of 12 months in a year, or perhaps because several crucial fractions, such as $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, of 12 can be taken painlessly. Thus the day ended with 24 hours. The **minute** and the **second** are Babylonian in nature, and 60 was their base (see the next chapter).

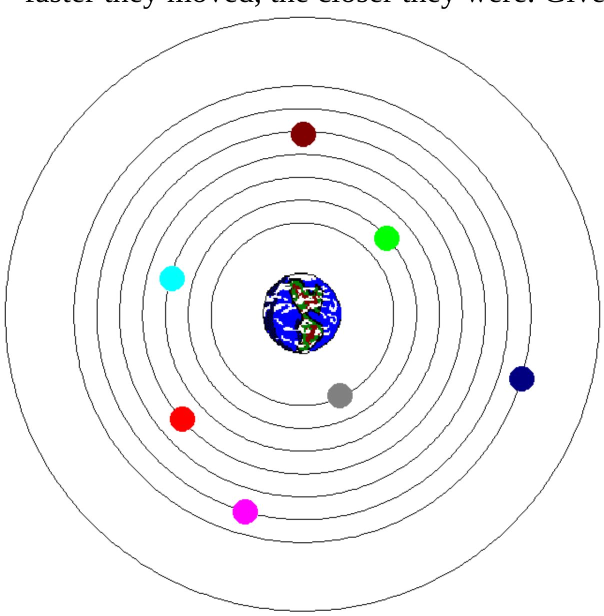
Finally, we arrive at the **week**. The week of seven days is far from universal across the planet. To the Romans, it was 8 days. To the Revolutionary French, it was 10 days. But in our society it consists of 7 days. **Why 7**?

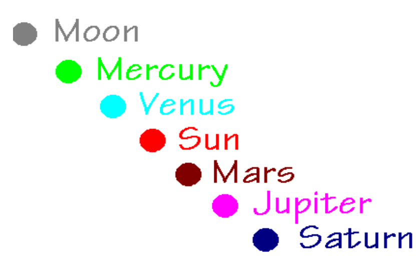
Could it be that that each week corresponds to a fourth of the lunar cycle? We have, of course, the Judeo-Christian tradition of a week, and certainly this is a major component for its survival to our times. Further back are the Babylonians, and the very ancient belief of the seven planets in the skies. As we mentioned in the Preface, from ancient times people had believed that there was the **Earth**, the fixed stars and seven wandering stars (or **planets**): **the Sun** ☉, **the Moon** ☽, **Mercury** ☿, **Venus** ♀, **Mars** ♂, **Jupiter** ♃ and **Saturn** ♄.

As we saw before, there was great deal of reluctance to add to the list of wandering stars. We saw in the Preface an argument that was launched against Galileo by a contemporary scholar. A more complete version of the argument is:

> There are seven windows in the head, two nostrils, two ears, two eyes and one mouth; so in the heavens there are two favorable stars, two unpropitious, two luminaries, and Mercury alone undecided and indifferent. From which and many others similar phenomena of nature such as the seven metals, etceteras, which it were tedious to enumerate, we gather that the number of planets is necessarily seven... Moreover, the satellites are invisible to the naked eye and therefore can have no influence on the earth and therefore would be useless and therefore do not exist. Besides, the Jews and other ancient nations, as well as modern Europeans, have adopted the division of the week into seven days and have named them after the seven planets. Now if we increase the number of planets, this whole and beautiful system falls to the ground.

Note the reference to the planets and the week. To be more specific, the belief from ancient times until the Copernican revolution was that the planets went around the Earth. Furthermore, they believed the distance from each planet to the Earth was inversely proportional to the planet's apparent velocity in the skies as viewed from Earth—the faster they moved, the closer they were. Given this, their placement of the planets was





The names of the days in English come from the Saxon, so the translation from the planets to the days of the week is not so obvious. Nevertheless, the Saxons had identified their gods and goddesses with Roman deities, and this identification makes the connection clearer. Even better, if one were to use one of the romance languages such as French, Italian or Spanish, the connection is apparent.

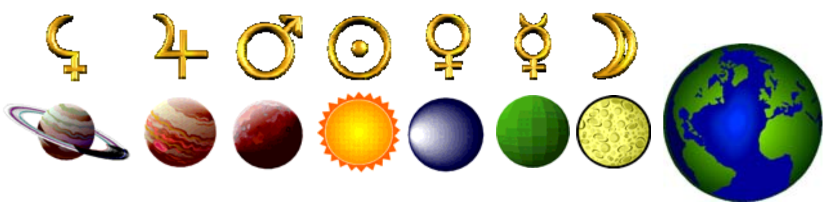
In the table, a list of the English and Spanish names for the days of the week together with their connections to the planets is given. As we saw above, each of the planets had a sign associated with it.

Planet	Day	Spanish	Saxon
Sun	Sunday	Domingo	Sun
Moon	Monday	Lunes	Moon
Mars	Tuesday	Martes	Tiw
Mercury	Wednesday	Miércoles	Woden
Jupiter	Thursday	Jueves	Thor
Venus	Friday	Viernes	Frigg
Saturn	Saturday	Sábado	Saturn

But if the order of the planets in the skies was **Saturn**, **Jupiter**, **Mars**, the **Sun**, **Venus, Mercury** and the **Moon**, how do we arrive to our present order for the days of the week? To understand this last ingredient to the puzzle, we have to be aware that each hour of the day was associated with a deity, and that the day was named after the god or goddess assigned to the first hour of that day. Suppose then our day starts with an hour of the Sun. Let us look to which deity each of the following hours will it be dedicated to:

1	2	3	4	5	6	7	8	9	10	11	12
Sun	Venus	Mercury	Moon	Saturn	Jupiter	Mars	Sun	Venus	Mercury	Moon	Saturn
13	14	15	16	17	18	19	20	21	22	23	24
Jupiter	Mars	Sun	Venus	Mercury	Moon	Saturn	Jupiter	Mars	Sun	Venus	Mercury

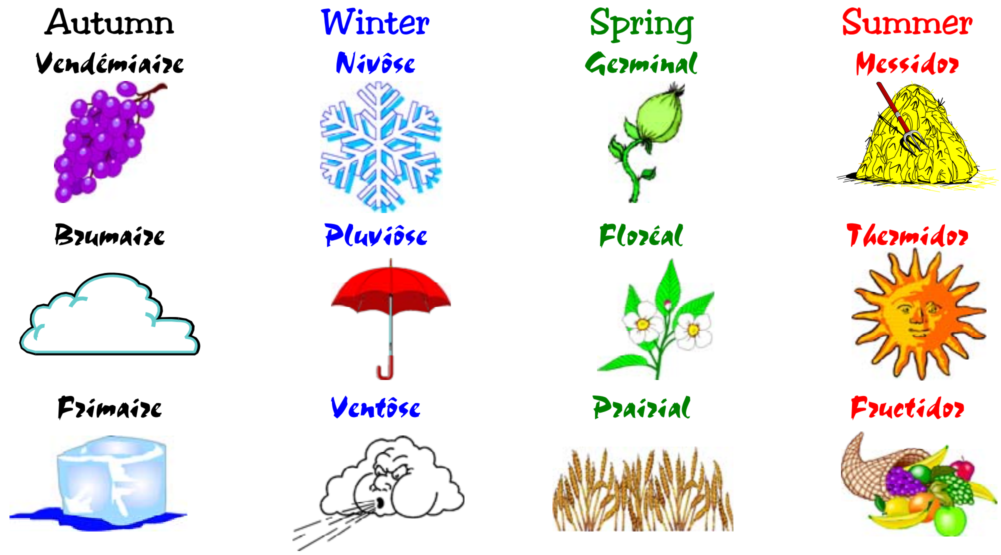
So **the first hour of the next day would be dedicated to the Moon**, thus, **Monday follows Sunday**. If we repeat this with each of the following days we get our traditional order for the days of the week!



We cannot leave this entertaining section on the Calendar without discussing briefly a more modern attempt to modify time measurement. One of the last major attempts to modify all the units of time occurred during the **French Revolution**—and while they succeeded with their changes concerning measurement for distance and mass, namely the **Metric System**, their attempts at time measurement modification lasted but a few decades.

Their changes were indeed radical: the year was to consist of 12 months of 30 days each (with 5 special days similar to the Mayan tradition). The month would consist of three weeks each of 10 days. The National Convention (who took over the running of the government) declared September 22, 1792 (the fall equinox) to be the first day of Year 1 of the Republic of France. What perhaps would be one of the ingredients that would inevitably doom the system were the names of the months, all which would be connected with weather or agricultural conditions, with no ties to the Romans or to the Pope.

The French Revolutionary Calendar



Note that the ending of the name of the month indicated the season. We now return to our main task at hand, that of looking at the deep roots of the earliest components of mathematics, number and shape, arithmetic and geometry.

Arithmetic

God made integers, all else is the work of mankind—
Kronecker

One could perhaps improve on the nineteenth century German mathematician's saying by exchanging the word **integers** by the words **counting numbers**. Let us start by reviewing briefly the complex history of the different types of numbers that inhabit our modern zoo.

As aforementioned, the first numbers to occur are naturally the **positive integers**, also called the **counting numbers**, or the **natural numbers—without a zero**. The idea of such numbers as mentioned above is old, and different cultures have had very different notations for representing these, the most basic of all mathematical ingredients. We have seen on such notation already, and we will briefly study two others later on.

It is impossible to tell when these numbers first appeared, but they are much older than any historical records. As mentioned above, recent evidence suggests that the first symbolic writing of mankind may have been to indicate counting. Additionally, the management not only of numbers, but also of very large numbers is old. We mention the Incas of South America who kept great accounting records of a large empire without ever developing a written language!

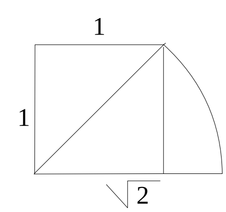
Some may be surprised to learn that the **zero** comes about roughly fifteen hundred years ago—thousands of years after the positive integers. One reason may be that there was no need for such a concept, and much less for a symbolic representation of it. Our zero comes from India, although as we saw before, the Mayans also had the concept.

$\mathbb{N}$ is the set of **counting numbers,** a very natural, intuitive set. We will agree to include 0, $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$, although historically that is far from essential. The notation we use to indicate collections of numbers is very common in mathematics courses, but some of its use is limited to such courses. Most of the notation comes from nineteenth century Germany. $\mathbb{N}$ could be for number, but it could also be for natural. As observed above, the human intuition of these numbers comes from very ancient times, but that is not the case for the marvelous notation we use to describe them.

We refer to our numeral system as **Hindu-Arabic**, after the two cultures from which we inherited it in the late Middle Ages. We have already encountered two of the most crucial ingredients to our numeral system: **base & position**. The notion of base is ancient, and as in our system, base ten is among the most common through various cultures, although base 2, base 5, base 12 and base 20 also occur in one form or another, as we saw with the Mayans. The more subtle power is in the position idea, namely that 15 does not mean the same as 51—which although harder to learn, is much more powerful in the long run. We will discuss this idea further later on.

The next collection of numbers to appear in the human landscape, the **positive rationals**, lie but a small jump from the positive integers. They also occur in pre-history, and they are easily justified as a change-of-units idea. Hence, when we say $\frac{2}{3}$, we call 2 the **numerator**, or counter, and 3 the **denominator**, or labeler. Our notation for the ratio of two positive integers is not ancient, and should not be taken for granted. Both the Egyptians and the Greeks would have benefited from such notation. There is no traditional symbol to denote the collection of positive rationals, although $\mathbb{Q}^+$ could be used—see below.

The Babylonians, more than 3,000 years ago, introduced something similar to our **decimal** notation, 3.14159 for example, except that their base was 60, and they are called **hexagesimals**. Hexagesimals will be one of the prevalent notations in Europe until the modern version of decimals is introduced roughly 400 years ago. We will look at hexagesimals in more detail in the next chapter.



The harsh realization that rational numbers were not enough to express all ideal lengths came early in the history of mathematics (more than 2000 years ago). Namely, if one wants to associate a number with each point on the line, then there are some natural lengths that aren't rational. In particular, as in the picture, the diagonal of a square of side 1. And thus **irrational** numbers were born. Among the earliest ones are $\frac{1+\sqrt{5}}{2}$, and $\sqrt{2}$, which is the length of the diagonal in the picture. So in order to correspond a number to each point of the line, the real numbers were created. But that does not really occur until the seventeenth century. However, satisfactory understanding of the real number system does not occur until the nineteenth century! One way to think of the real numbers: **real numbers are points of the line.**

Negative numbers are even more recent, and although operating **in the red** appears early in the history of Western accounting (14th century), and even earlier in the Chinese culture, negative numbers are not universally accepted even by the end of the seventeenth century. Brilliant minds—including **Pascal** and **Descartes**—would dispute their need or their existence as late as the 1600's. To Descartes, one of the creators of coordinate axes, there would be only one quadrant of what we refer to as the Cartesian plane, the first quadrant. He was so opposed to them he even referred to negative numbers as imaginary. Later on, many would have problems with what **a minus times a minus** is in the eighteenth century.

Once negative integers are accepted, then we have the **integers**, or **whole numbers.** One uses $\mathbb{Z}$ for this collection, $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. $\mathbb{Z}$ is for *zahlen*, which means numbers in German. And then all **rational numbers**, positive and negative is denoted by $\mathbb{Q}$. $\mathbb{Q}$ could stand for quotient, but it could also be conjectured that $\mathbb{Q}$ comes before $\mathbb{R}$. And $\mathbb{R}$ is the set of **real numbers**, the real line.

Those pesky numbers that most people ignore, those embarrassing objects called **complex numbers** started appearing in Europe as early as the 1500's. But, it was not until the 1800's that they became widely accepted by the mathematical community. And by the 20th century, they have become necessary for engineering. $\mathbb{C}$ denotes the set of complex numbers.

Finally, **hypernumbers** (now more commonly referred to as **matrices**) appear in the middle part of the nineteenth century, and become overwhelmingly important in the second half of the 20th century.

The following table summarizes the history of the different components of the number system:

NUMBERS THROUGH the AGES		
From Oldest to Youngest	Years Ago[1]	**Where[2]**
Counting Numbers: 1,2,3,4,5,6,...	At least ten thousand years	**Everywhere?**
Positive Fractions: $\frac{1}{2},\frac{1}{3},\frac{2}{3},\ldots$	Old: 10,000?	**Everywhere?**

[1]These times are grossly simplified.

[2]Very rough geographical approximations. And certainly many ideas have flourished in many places if not simultaneously, independently, of each other.

Some Positional Notation Hexagesimals: $\frac{1}{60}, \frac{1}{60^2}, \ldots$	4,000	**Mesopotamia**
Some Irrationals: $\sqrt{2}, \frac{1+\sqrt{5}}{2}, \sqrt[3]{2}, \pi \cdots$	2,500	*Greece*
0 and Positional Notation	1,500	India
Negative Numbers: $-1, -2, -\frac{1}{2}, \ldots$	500	**Europe**
Decimals & Real Numbers: 2.46, e, $\ln(2)$	350	**Europe**
Imaginary & Complex Numbers: $i, 1+i, \sqrt{i}, \ldots$	200	**Europe**
Hypernumbers & Matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots$	100	**Europe**

We end our section concerning arithmetic with a deeper, but short, discussion of the real numbers and complex numbers. Another way to think of the real numbers is: **real numbers are arbitrary decimals**, for example: 0.101001000100001000001 is a real number.

The criterion for a decimal to be a rational number is well known: **a decimal is rational if it is eventually periodic**, in other words, **some string of digits repeats indefinitely**. For example, 1.23333333… is rational since 3 is a repeating block, and indeed 1.23333333…$=\frac{37}{30}$. Similarly, $17.714285714285\overline{714285}\cdots$ is also a fraction, $\frac{124}{7}$.

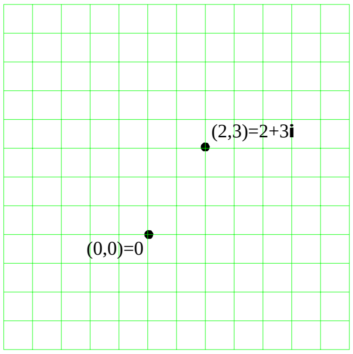
The fundamental property that the real numbers satisfy, that the rationals do not, is the following:

> if $x_1 > x_2 > x_3 \cdots$ is an infinite sequence of positive real numbers, then there is necessarily a real number x to which this sequence is converging, that is, the limit of the sequence exists: $\lim_{n\to\infty} x_n = x$ for some real number x.

This principle, not clearly enunciated until the 1800's, is one of the building blocks of modern mathematical analysis.

In the same way as $\sqrt{2}$ was necessary to solve $x^2 = 2$, the complex numbers were necessary to solve $x^2 + 1 = 0$. We let $\mathbf{i}$ be such a solution: $\mathbf{i}^2 = -1$. Then we begin to do arithmetic: $3\mathbf{i}$ is just $\mathbf{i}+\mathbf{i}+\mathbf{i}$, and if we add 2 to it we get $2+3\mathbf{i}$, and this is a typical complex number, and it is as complex as one needs to get. Suppose we want to add 1 to $2+3\mathbf{i}$, it's easy: we get $3+3\mathbf{i}$. Similarly, if we want to add $\mathbf{i}$ to $3+3\mathbf{i}$, we get $3+4\mathbf{i}$. In short, $(2+3\mathbf{i})+(1+\mathbf{i}) = 3+4\mathbf{i}$.

What about multiplication? Just use the distributive law: $(2+3\mathbf{i})(1+\mathbf{i}) = 2(1+\mathbf{i}) + 3\mathbf{i}(1+\mathbf{i}) = 2+2\mathbf{i}+3\mathbf{i}+3\mathbf{i}^2$. The only confusing term is $3\mathbf{i}^2$, but recalling what $\mathbf{i}$ is all about, $\mathbf{i}^2 = -1$, so $3\mathbf{i}^2 = -3$, hence $(2+3\mathbf{i})(1+\mathbf{i}) = -1+5\mathbf{i}$.



Furthermore, what is very important, but yet not often stressed enough, is that complex numbers have as much a geometrical reality as real numbers. Just as every real number corresponds to a point in the line, each complex number corresponds to a point in the plane.

We have, for example, $2+3\mathbf{i}$ corresponding to the point whose Cartesian coordinates are 2 and 3 respectively.

So the *x*-axis corresponds to the real numbers while the *y*-axis are the pure imaginary, which are numbers of the form $\mathbf{i}, 2\mathbf{i}, -\mathbf{i}$, etcetera.

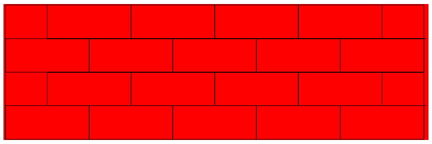
We now consider the historical roots of the other ancient branch of mathematics:

Geometry

God ever geometrizes—*Plato*

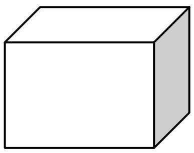
Together with number, shape runs deep in the history of mankind. The roots of geometry are also very old (perhaps even older than those of arithmetic). And the oldest geometrical questions had to do, again with measurement. This time of space, thus the notions of **length, angle, distance, area** and **volume** are among the first to occur.

Of course, length is very much associated with counting. Once we have a unit of linear measure, we count how many times it fits around the room, or whatever we are attempting to take the length of, and we have arrived at an estimate. Of course, fractions occur naturally in this context.



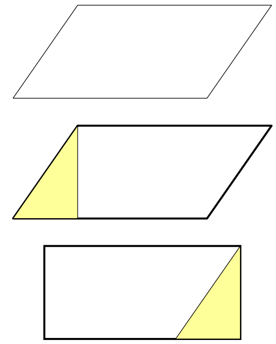
One can only use common sense to speculate what did happen, and one can easily conjecture that the oldest **area** to be computed was the **rectangle**: the **base**×**height** formula for the area could easily be deduced via multiplication from brick laying or tiling examples.

Naturally, we use the contemporary language of formulas to express this idea; however, it is good time to remind ourselves **that formulas are not the only way of expression or communication.** There are other forms to express the result for the area of a rectangle such as **two rectangles with the same base and height have equal areas.**



Similarly, the first **volume** to be achieved was, most probably, that of a rectangular

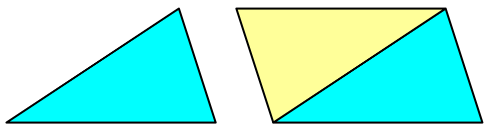
parallelepiped (a rectangular box) with the well-known **length×width×height** expression for its volume.

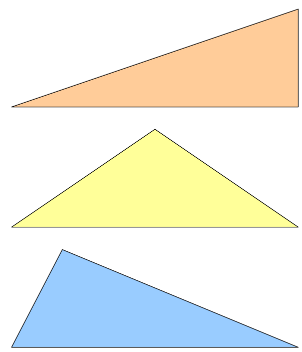


The next area, after the rectangle, to be computed was, probably, that of a **parallelogram**, which is also **base × height**) That this was done early follows from the easy rearrangement of any parallelogram into a rectangle.

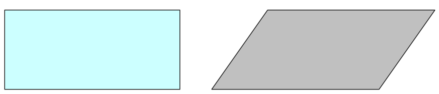
And then the **triangle** could not be far behind since two of them make a parallelogram:

$$\text{Area} = \tfrac{1}{2}(\text{base} \times \text{height})$$





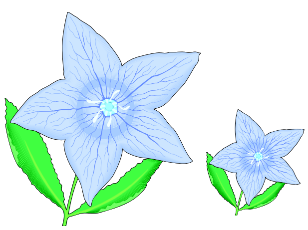
The more sophisticated idea of **gliding a vertex of a triangle** (or the side of a rectangle), so that the area does not change as we glide, was probably more recent. Thus, **the three shaded triangles all have the same area because they have the same base and the same height.**



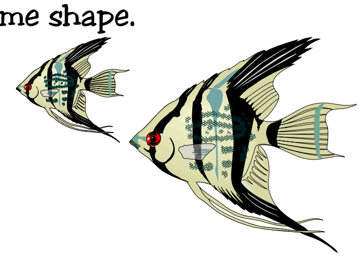
Whether the idea was applied first to triangles or to parallelograms is impossible to tell.

Another fundamental notion in early geometry is that of **similarity**.

Two figures are similar if they have the same shape.



The need and realization for such a notion arises from religious ornaments and artifacts, and its prevalent occurrence in nature:



It is not clear whether ancient civilizations completely understood the relation between the area (or volume) of a figure and its linear dimensions.

Thus, for example, **when the linear dimensions are doubled, the area is quadrupled. Similarly, if the linear dimensions are halved, then the volume is one eighth of the original volume**. How that fundamental fact was first discerned is, probably, due again to the tiling example, as the figure shows.