Homework #9

❶ Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. a, b, c are positive integers.

① If a and b are divisors of c, then $a+b$ is a divisor of c.
② If a and b are multiples of c, then $a+b$ is a multiple of c.
③ If a and b are relatively prime, then so are ab and $a+b$.
④ For every positive integer n, $6^n \equiv 6 \bmod 10$.
⑤ 63000 has exactly 24 odd divisors.

❷ In all the following situations, the balls are colored but the buckets are not.

① In how many ways can you break 4 balls into 2 buckets so that each bucket gets two balls?
② In how many ways can you break 6 balls into 2 buckets so that each bucket gets three balls?
③ In how many ways can you break 6 balls into 3 buckets so that each bucket gets two balls?

❸ In order to advertise the movie E.T., a theater hires 16 children to hold the letters of the word EXTRATERRESTRIAL.

① In how many ways can the children be assigned letters to hold?
② How many anagrams of the word are there if all the vowels are together?
③ How many anagrams are there if all the vowels are together and in alphabetical order?
④ How many anagrams are there if all the vowels are in alphabetical order?
⑤ How many anagrams are there if all the vowels are in alphabetical order and all the consonants are in alphabetical order too?
⑥ How many anagrams are there if all the vowels are together and in alphabetical order, and so are the consonants?
⑦ How many anagrams are there if all the letters are in alphabetical order?

❹ Give the coefficients of each of the required terms in the expansion of $(x+y+z+w)^{14}$:

① x^{14} ② $x^5y^4z^3w^2$ ③ $x^6y^4z^3w^2$ ④ $x^5y^3z^2w^4$

❺ As an experiment in arithmetic, the 112 pupils of the Mena School are each given a card with one side yellow and the other side green, and both sides numbered with the same number, between 1 and 112; each receiving a distinct number. At the start all pupils are to show the yellow side of the card, and a caller is going to consecutively call numbers starting at 1 and ending with 112. If a student is holding a card that is a multiple of the number called, the student then flips the card to the other side—from green to yellow or from yellow to green as the case may be. At the end, which cards show green?

② What is the answer to the question if the caller calls all the numbers but not necessarily in order?

③ Suppose the game is played so that once the card shows green it is never turned back. What is the minimum number of calls (in any order) he has to make to have exactly only one yellow card showing?

Bonus: Explain without much algebraic manipulation why the following identity holds:

$$\sum_{k=0}^{j}\binom{30}{k}\binom{20}{j-k}=\binom{50}{j}, \text{ for any } 0\le j\le 20.$$

(**Hint:** Start by fixing a j and thinking about what you are doing on the right hand side, and then try to use the first counting principle on the left hand side.)

Proofs of the Week

1. If $x_1, x_2, \ldots, x_n \in \mathbf{P}$, then $x_1 x_2 \cdots x_n \in \mathbf{P}$. *the product of positives is positive*

2. If $x, y \in \mathbf{N}$, then $x+y \in \mathbf{N}$. *sum of negatives is negative*

3. If $x, y \in \mathbf{N}$, then $xy \in \mathbf{P}$. *product of negatives is positive*

4. If $x \in \mathbf{P}$ and $y \in \mathbf{N}$, then $xy \in \mathbf{N}$.

5. If $x \neq 0$, then $x^2 \in \mathbf{P}$. *squares are positive*