Homework #12

❶ Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

① A relation that is symmetric and transitive is automatically reflexive.

② The last two digits of any number that ends in a 3 when raised to the 40^{th} power are 00.

③ 933354738245187864 is a multiple of 36.

④ $456738937\,456738937^3$ ends in a 3.

⑤ Let a, b and m be positive integers. If m divides ab, then either a is a multiple of m or b is a multiple of m.

❷ We are considering posets with 12 elements. Give the largest possible answer for each of the following and justify it by giving an example:

① How many minimal elements?

② How many minimum elements?

③ How many maximal elements?

④ How many maximums?

⑤ Is it possible to have such a poset, which has exactly 7 minimal elements and 8 maximal ones? Give reasons or an example.

❸ For a given *n* let P_n, Q_n and S_n stand for the following sets of $n \times n$ (0,1)-matrices: a matrix in P_n has exactly one 1 in each row and in each column; a matrix in Q_n has exactly one 1 in each row; a matrix in S_n has exactly one 1 in each row and it is symmetric.

① Among these three sets decide which is contained in which.

② $|P_n| = ?$

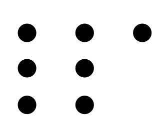
③ $|Q_n =| ?$

④ Compute $|S_1|$, $|S_2|$, $|S_3|$, $|S_4|$, $|S_5|$, $|S_6|$

⑤ Give a recursion connecting the $|S_n|$'s.

❹ Consider the expression $x^{2^{y^3}} y^2 - \frac{5}{4-x}$. Write its tree and transverse it in all three methods.

❺ Consider the partition 3+2+2 of 7. One way to visualize this partition is to draw a diagram representing it:



by drawing lined-up dots as many in a row as necessary. The **dual** partition is the one associated with the columns of this diagram: 3+3+1.

Another example: the dual of 7: ● ● ● ● ● ● ●
is of course, 1+1+1+1+1+1+1.

① Find the dual of each of the partitions of 7.

② Do you notice anything about the ordering and duality? There is something to observe.

③ Find the dual of each of the following partitions of 15:

6+5+4 5+4+3+2+1 4+4+4+3 3+3+3+3+3

Bonus: A partition of n is called **self-dual** if it equals its dual (see previous exercise). Let $D(n)$ be the number of self-dual partitions of n. Compute $D(n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.

Proofs of the Week

1. If $x > 0$ and $x < 1$, then $x^2 < x$.

2. If $x < 0$, then $x^3 < 0$.

3. If $x > 1$, then $0 < x^{-1} < 1$.

4. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$. **Reciprocation**