

# HOMEWORK #1

- ❶ Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let  $u = a + bi$  and  $v = c + di$  be complex numbers where  $a, b, c, d \in \mathbb{R}$ .

- ① The conjugate of a sum is the sum of the conjugates:  $\overline{u + v} = \bar{u} + \bar{v}$ .
- ② The conjugate of a product is the product of the conjugates:  $\overline{u \times v} = \bar{u} \times \bar{v}$ .
- ③ One can find exactly two complex numbers  $x$  such that  $x^2 = i$ .
- ④ It is more likely that one tosses exactly 3 heads in 6 tosses of a coin than for one to toss exactly 5 heads in 10 tosses of a coin.
- ⑤ Since  $1 + i$  is a root of the polynomial  $x^3 - (4 - i)x^2 + (10 - 4i)x - 10$ , so is  $1 - i$ .

- ❷ Let  $u = 3 + 4i$  and let  $z = -1 + 2i$ . Compute all of the following:

- |                        |                           |                        |
|------------------------|---------------------------|------------------------|
| ① $u + z$              | ② $u - z$                 | ③ $uz$                 |
| ④ $\frac{z^2}{z + u}$  | ⑤ $\frac{\bar{z}}{z + u}$ | ⑥ $\frac{\bar{u}}{zu}$ |
| ⑦ $\frac{\bar{z}}{zu}$ | ⑧ $\frac{\bar{z}}{z + u}$ | ⑨ $\frac{\bar{u}}{zu}$ |
| ⑩ $\frac{\bar{z}}{zu}$ | ⑪ $z \div u$              | ⑫ $u \div z$           |

- ❸ Use polar coordinates to find all solutions to  $z^4 = -16$ . Express your answers both in Cartesian and polar coordinates.

- ❹ We remarked in the text that conditionals are the most common claims in mathematics and science (probably life), but they certainly don't always come in the neat form: **if ..., then ...**. Below are some sentences in English. Each of them is equivalent to either

- ❶: **If I go to the movies, then I cry**, or its converse
- ❷: **If I am crying, then I am at the movies**.

Decide which is which.

- ① Only if I go to the movies, do I cry.
- ② Only if I cry, do I go to the movies.
- ③ Unless I cry, I don't go to the movies.
- ④ Unless I go to the movies, I don't cry.
- ⑤ It suffices for me to go to the movies in order for me to cry.
- ⑥ It suffices for me to cry in order to go to the movies.
- ⑦ It is necessary for me to go to the movies in order for me to cry.
- ⑧ It is necessary for me to cry in order for me to go to the movies.

- ❺ The following statement is true for any positive integer  $n$ :  
If  $2^n + 1$  is prime, then  $n$  cannot have any odd divisors bigger than 1.

- ① Test the statement for  $n = 1, 2, 3, 4, 5, 6$ .

- ② Does the statement justify any claims about the primeness of  $2^{20} + 1$ ? Explain.
- ③ Does the statement justify any claims about the primeness of  $2^{32} + 1$ ? Explain.
- ④ Do you think the converse of this statement is true? Explain.

## PROOFS OF THE WEEK

We will start with some notation and some assumptions. But first we have two rules of common sense (and common inference):

**Common Sense Rule #1.** If a statement is known for every  $a$ , then one can infer that it is known for any specific  $a$ —in other words one can choose (or select) the element to apply the sentence to.

**Warning:** In contrast if something is known for some  $a$ , then we are not free to choose to apply that sentence to a given specific  $x$ . For example, if it is known that  $\forall a f(a) = 7$ , then we know that  $f(1) = 7$ . On the other hand, if all we know is that  $\exists a f(a) = 7$ , then we cannot conclude that  $f(1) = 7$ .

**Common Sense Rule #2.** If we know that two things are equal, then we can freely substitute one for the other in any well-defined expression, and the two shall be equal too.

For example, if  $a = b$  is known, then we know that  $\frac{a+x}{y} = \frac{b+x}{y}$ .

We will let  $\mathbb{R}$  denote the set of **real numbers**, which we will assume as given. We will also assume that we know how to add and multiply two of them, namely if  $a, b \in \mathbb{R}$ , then there are uniquely determined real numbers given by the sum of  $a$  and  $b$ ,  $a + b$ , and by their product,  $a \times b$ .

## ARITHMETIC

We will assume the following facts<sup>1</sup> about these operations:

<b>A1</b>	<b>Addition is commutative.</b>	$\forall a \forall b \ a + b = b + a$
<b>A2</b>	<b>Addition is associative.</b>	$\forall a \forall b \forall c \ a + (b + c) = (a + b) + c$
<b>A3</b>	<b>0 is the identity.</b>	$\forall a \ a + 0 = a$
<b>A4</b>	<b>Additive inverses exist.</b>	$\forall a \exists b \text{ such that } a + b = 0$

Things to be proven:

1.  $0 + 0 = 0$

**Idempotency of the identity.**

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<sup>1</sup> Assumptions will always be boxed, and things to be proven are numbered.

2.  $\forall a \forall x \forall y$ , if  $a + x = a + y$ , then  $x = y$ .      **Cancellation.**
3.  $\forall a$ , if  $a + a = a$ , then  $a = 0$ .      **Uniqueness of idempotent.**
4.  $\forall a \exists! b$  such that  $a + b = 0$ .      **Uniqueness of inverse.**

The symbol  $\exists!$  reads, there is exactly one. One proves uniqueness usually by taking two things that satisfy something, and proving that they are actually equal.

**Definition.** The unique  $b$  in 4 is then called  $-a$ , the negative of  $a$ . Define **subtraction** by  $a - b = a + -b$ .