Homework # 4

❶ Consider the veracity or falsehood of each of the following statements, and argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size.

① If $\mathbf{AB}$ is nilpotent, then so is $\mathbf{BA}$.

② If $\mathbf{A}$ and $\mathbf{B}$ are nilpotent, then so is $\mathbf{A}+\mathbf{B}$.

③ If $\mathbf{A}$ is nilpotent, then $\mathbf{I}-\mathbf{A}$ is invertible.

④ If $\mathbf{A}$ is nilpotent of index 4, then $\deg\mu_{\mathbf{A}+2\mathbf{I}}(x)=4$.

⑤ The centralizer of a 2×2 matrix may have dimensions 2 and 4 only.

❷ Find a basis for the centralizer of $\begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}$.

❸ **On Arithmetic Progressions.** A row is said to be an **arithmetic progression** if the difference between any entry and its preceding entry is constant throughout the row. $(1 \quad 3 \quad 5)$ is such an example since $3-1=5-3=2$. Let $\mathbf{P}_n$ be all $1\times n$ matrices that are an arithmetic progression. Show $\mathbf{P}_n$ is a linear space and find its dimension by exhibiting and proving you have a basis.

❹ For every $n\geq 1$, let $\mathbf{G}_n$ be the graph whose adjacency matrix is $\mathbf{N}_n+\mathbf{N}_n^{\mathrm{T}}$ where $\mathbf{N}_n$ denotes the basic nilpotent matrix. Do the following:

① Show $\mathbf{G}_n$ is bipartite.

② Find the minimum polynomial for $\mathbf{G}_n$ for $n=1,2,3,4,5,6$. Hint: Use the **Polynomial of a matrix** spreadsheet in the website.

③ Can you observe something special on the roots of these polynomials?

Bonus: Give a description of the minimum polynomial of $\mathbf{G}_n$ for arbitrary n, and this could include a recursion.

❺ **On Minimum Polynomials.** Let $\mathbf{M}$ be a square matrix.

① Let $p(x)$ be an irreducible polynomial, and assume that $p(\mathbf{M})$ is not invertible. Show that $p(x)$ is a factor of $\mu_{\mathbf{M}}(x)$.

② Let $\mathbf{C}\neq\mathbf{O}$ be a column, and assume that $\mathbf{C},\mathbf{MC},\mathbf{M}^2\mathbf{C},\ldots,\mathbf{M}^{k-1}\mathbf{C}$ are linearly independent, but $\mathbf{M}^k\mathbf{C}=a_{k-1}\mathbf{M}^{k-1}\mathbf{C}+\cdots+a_1\mathbf{MC}+a_0\mathbf{C}$. Show the polynomial $q(x)=x^k-a_{k-1}x^{k-1}-\cdots-a_1x-a_0$ has an irreducible factor in common with $\mu_{\mathbf{M}}(x)$. Hint: Use ①.

Let $\mathbf{M}=\begin{pmatrix} -68 & 9 & 50 & 30 & 18 & -38 \\ -49 & 7 & 36 & 22 & 12 & -27 \\ -77 & 10 & 57 & 33 & 20 & -42 \\ -94 & 12 & 69 & 41 & 24 & -51 \\ -107 & 14 & 78 & 45 & 29 & -58 \\ -110 & 14 & 81 & 46 & 28 & -58 \end{pmatrix}$.

③ Multiply $\mathbf{MJ}_{6\times1}$ and use this product to find a factor of $\mu_{\mathbf{M}}(x)$.

④ Then proceed to multiply by another column to find another factor.

Use the **Polynomial of a matrix** program to keep checking whether you have the minimum polynomial or not.

⑤ Repeat until you have the minimum polynomial.