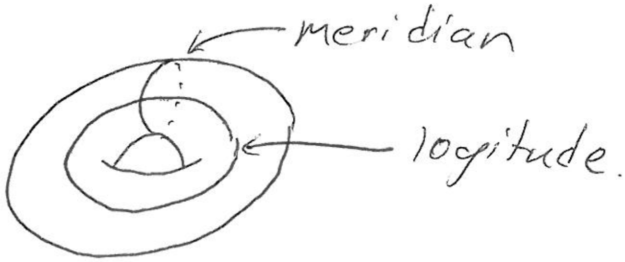
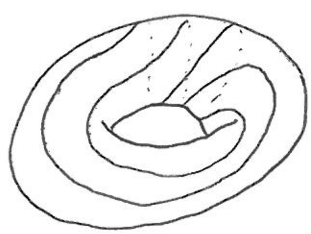
Hyperbolic Knot Boot Camp.

<u>Def</u>] A <u>torus knot</u> is any knot in S^3 that is abient isotopic to ~~the~~ a simple closed curve on the standard torus embedded in S^3.

These knots are often denoted (p,q) torus knots where p indicates how many times the knot wraps around in the meridinal direction and q represents how many times the knot wraps around in the longitudinal direction.



Example



$(3,2)$ torus is the trefoil

<u>Satellite knots</u>

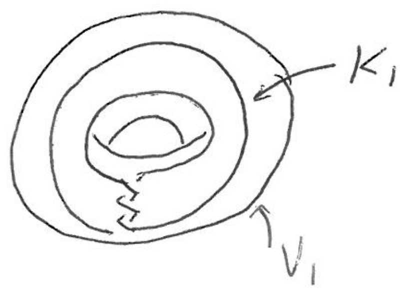
Let K_1 be a knot contained inside a solid torus V_1. Let K_2 be a knot in S^3. Let V_2 be a closed regular nbh of K_2. Replace V_2 with V_1 s.t. the boundary of a meridian disk of V_1 is mapped to the boundary of a meridian disk of V_2. Let K_3 be the image of K_1 under this replacement

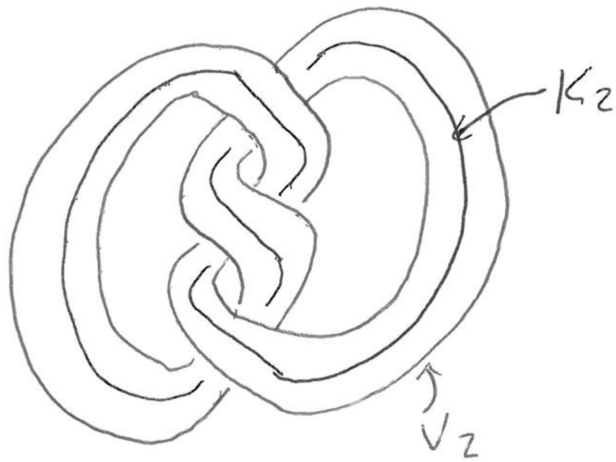
K_3 is a satellite knot

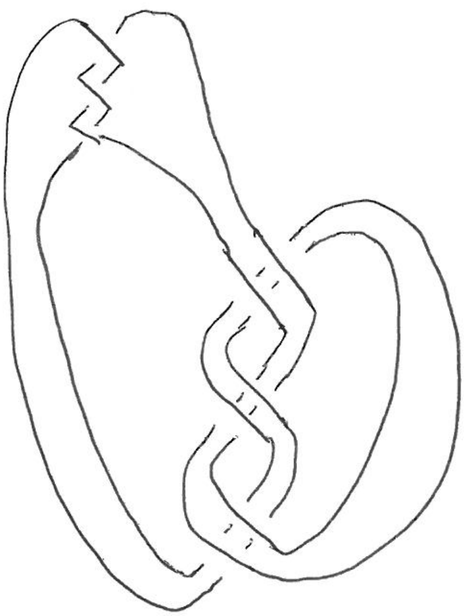
K_2 is a companion knot

K_1 is the pattern knot

∂V_2 is the companion torus in $S^3 - K_3$.







K_3

Fact: Every connected sum $K_1 \# K_2$ is a satellite knot with pattern K_1 and companion K_2.

Thm (Thurston)

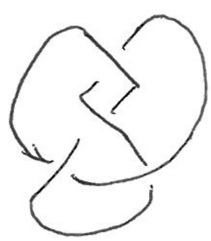
Every knot fits into exactly one of the following categories

1) Torus knots
2) Satellite knots
3) Hyperbolic knots.

Thm (Menasco)

Let K be a prime, alternating knot that is not a $(2,n)$ torus knot, then K is hyperbolic.

Ex



is hyperbolic

Def $K \subset S^3$ is a hyperbolic knot if $S^3 - K$ has a complete Riemannian metric of constant curvature -1.

From last time

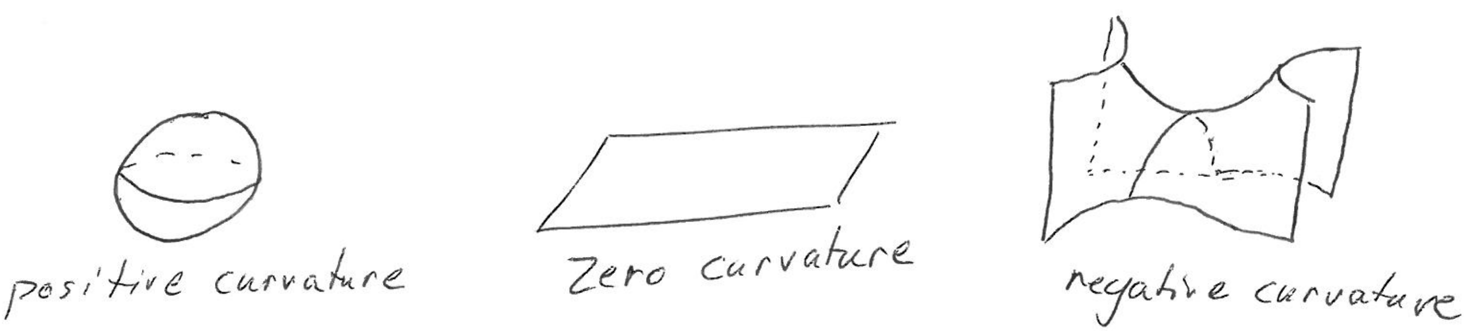
Def A top. space X is an n-manifold if X is hausdorff, 2nd-countable and locally homeomorphic to $\mathbb{R}^n$.

Examples 

S^3, $S^3 - K$

Recall **Def** A 3-manifold is hyperbolic if it has a complete Riemannian metric of constant curvature -1.

What does this mean in dimension 2



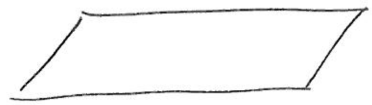
The simplest hyperbolic 3-manifold

$\mathbb{H}^3 = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq 1 \}$ with a geodesic metric s.t. geodesics in $\mathbb{H}^3$ are subarcs of circles that meet $\partial \mathbb{H}^3$ in right angles or straight line segments that pass through the origin

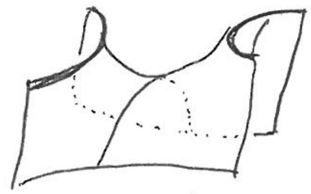
What does this definition mean?

In 2D:

positive curvature



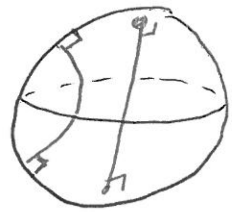
zero curvature



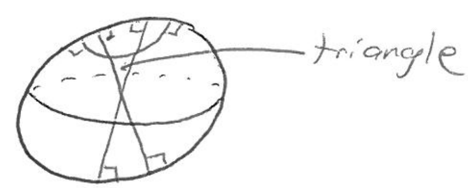
negative curvature

The simpilest hyperbolic 3-manifold

$\mathbb{H}^3 = \{(x, y, z) \mid x^2+y^2+z^2<1\}$ with a metric s.t. geodesics in $\mathbb{H}^3$ are straight line segment that pass through the orgin or are subarcs of circles that meet $\partial\mathbb{H}^3$ in right angles.



A hyperbolic triangle in $\mathbb{H}^3$ has edges consisting of subarcs of geodesics



Fact| The sum of the angles of a hyperbolic triangle is always less than $180°$

Fact| The area of a hyperbolic triangle is $\pi - (\alpha+\beta+\gamma)$ where α, β, γ are the angle measures in radians.

A hyperbolic tetrahedra has edges consisting of subarcs of geodesics in $\mathbb{H}^3$ and faces consisting of portions of planes and spheres that meet $\partial \mathbb{H}^3$ in right angles.

hyperbolic tetrahedra.

Fact All hyperbolic tetrahedra have finite volume.

Def A hyperbolic 3-manifold is the union of finitly many hyperbolic tetrahedra glued together allong faces via isometries (homeomorphisms with no metric distortion).

Def A hyperbolic knot is a knot s.t. $S^3 - K$ is a hyperbolic 3-manifold.

Def Given a hyperbolic knot K, the sum of the volumes of all hyperbolic tetrahedra in $S^3 - K$ is the volume of K.

Thm (Mostow Rigidity) If $S^3 - K_1 \cong S^3 - K_2$, then K_1 and K_2 have the same volume.

Ex Vol() $\approx 2.02988321\ldots$

Additional topics

- Volume is preserved under mutation
- Lackenby's bound on volume for alternating knots.